

New Algebraic Approaches to Classical Boundary Layer Problems¹

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Abstract

In this paper, we use various ansatzes with undetermined functions and the technique of moving frame to find solutions with parameter functions modulo the Lie point symmetries for the classical non-steady boundary layer problems. These parameter functions enable one to find the solutions of some related practical models and boundary value problems.

1 Introduction

In 1904, Prandtl observed that in the flow of slightly viscous fluid past bodies, the frictional effects are confined to a thin layer of fluid adjacent to the surface of the body. Moreover, he showed that the motion of a small amount of fluid in this boundary layer decides such important matters as the frictional drag, heat transfer, and transfer of momentum between the body and the fluid. The two-dimensional classical non-steady boundary layer equations

$$u_t + uu_x + vu_y + p_x = u_{yy}, \quad (1.1)$$

$$p_y = 0, \quad u_x + v_y = 0 \quad (1.2)$$

are used to describe the motion of a flat plate with the incident flow parallel to the plate and directed to along the x -axis in the Cartesian coordinates (x, y) , where u and v are the longitudinal and the transverse components of the velocity, and p is the pressure. Lie

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point symmetries of the above equations were obtained by Ovsiannikov [O]. The three-dimensional classical non-steady boundary layer equations are:

$$u_t + uu_x + vv_y + ww_z = -\frac{1}{\rho}p_x + \nu u_{yy}, \quad (1.3)$$

$$w_t + uw_x + vw_y + ww_z = -\frac{1}{\rho}p_z + \nu w_{yy}, \quad (1.4)$$

$$p_y = 0, \quad u_x + v_y + w_z = 0, \quad (1.5)$$

where ρ is the density constant and ν is the coefficient constant of the kinematic viscosity. The Lie point symmetries of the equations (1.3)-(1.5) were obtained by Garaev [G] and by Vereschagina [V]. Lie group method is one of most important ways of solving nonlinear partial differential equations. However, the method only enables one to obtain certain special solutions. It is desirable to find more effective ways of solving nonlinear partial differential equations.

The aim of this paper is to develop more powerful methods than the Lie group method of solving the above classical non-steady boundary layer problems. Based our earlier work [X] on function-parameter exact solutions of the equation of nonstationary transonic gas flow discovered by Lin, Reisner and Tsien [LRT], and its three-dimensional generalization, we find ansatzes with various undetermined functions and obtained solutions with parameter functions modulo the Lie point symmetries for the above boundary layer equations. In the three-dimensional case, we have also used the technique of moving frame. Our results can be applied to certain related practical models and boundary value problems by specifying these parameter functions. Below we give a more detailed introduction.

Lin, Reisner and Tsien [LRT] found the equation

$$2u_{tx} + u_x u_{xx} - u_{yy} = 0 \quad (1.6)$$

for two-dimensional non-steady motion of a slender body in a compressible fluid, which was later called the “equation of nonstationary transonic gas flows” (cf. [M]). Using certain finite-dimensional stable range of the nonlinear term $u_x u_{xx}$, we obtain in [X] the following solution of the above equation:

$$u = \frac{x^3}{(\sqrt{3}y + \beta)^2} + \frac{\beta' x^2}{\sqrt{3}y + \beta} - \frac{2\beta''}{3}(\sqrt{3}y + \beta)x - \frac{4\beta'\beta''}{3}y^2 - \frac{2\beta'''}{27}(\sqrt{3}y + \beta)^3, \quad (1.7)$$

where β is an arbitrary function of t . Since the above solution blows up on the moving line $\sqrt{3}y + \beta = 0$, it reflects partial phenomena of gust. Moreover, we have found a family of solutions of the three-dimensional generalization of the equation (1.6) blowing

up on a rotating and translating plane $y \cos \alpha(t) + z \sin \alpha(t) = f(t)$, which reflect partial phenomena of turbulence. Our work [X] suggested some possible approaches to more general nonlinear partial differential equations in fluid dynamics, such as the classical non-steady boundary layer problems we will deal with in this paper.

By the divergence-free condition in (1.2), we write u and v in the potential form:

$$u = \xi_y, \quad v = -\xi_x, \quad (1.8)$$

where ξ is a function of t, x, y . It is known that the solution space of the equations (1.1) and (1.2) is invariant under the following transformation T (cf. [O]):

$$T(u) = u(t, x, y + \sigma), \quad T(v) = v(t, x, y + \sigma) - \sigma_t - \sigma_x u, \quad T(p) = p \quad (1.9)$$

for any function σ of t, x . Modulo this transformation, our main method of solving the equations (1.1) and (1.2) is to assume

$$\xi = f(t, x) + \kappa(t, x)y + g(t, x)H(t, y), \quad (1.10)$$

where f, κ, g are functions to be determined and H is a more sophisticated function that is either first given or is to be solved under certain conditions. The term $\kappa(t, x)y$ is used to make the function H more flexible. With the ansatz (1.10), the equation (1.1) is reduced to a system of several nonlinear partial differential equations in t, x . While by Lie group method, one usually reduces it to a single nonlinear partial differential equation in t, x . This in a way shows that our method is more powerful than the Lie group method. Imposing some conditions on the above undetermined functions, we are able to solve the system of partial differential equations. In this way, we obtain three families of function-parameter exact solutions of the two-dimensional classical non-steady boundary layer problem (1.1) and (1.2). For instance, we have the following solutions (see Theorem 3.2):

$$u = \frac{\beta''x}{2\beta'} + \frac{e^{-\beta}}{\sqrt{\beta'}} \Im \left(\frac{x}{\sqrt{\beta'}} \right) \cos(\sqrt{\beta'} y), \quad (1.11)$$

$$v = -\frac{\beta''y}{2\beta'} - \frac{e^{-\beta}}{\sqrt{(\beta')^3}} \Im' \left(\frac{x}{\sqrt{\beta'}} \right) \sin(\sqrt{\beta'} y), \quad (1.12)$$

$$p = \frac{(\beta'')^2 - 2\beta'\beta'''}{4(\beta')^2} x^2 - \frac{1}{2\beta'} e^{-2\beta} \left[\Im \left(\frac{x}{\sqrt{\beta'}} \right) \right]^2, \quad (1.13)$$

where β is an arbitrary increasing function of t and \Im is any one-variable function. Applying the symmetry transformations of the equations (1.1) and (1.2) (see next section) such as the one in (1.9), we can get solutions with more parameter functions.

The main contents of this paper are devoted to solving the three-dimensional classical non-steady boundary layer equations (1.3)-(1.5). Similarly, we write u, v, w in the potential form:

$$u = \xi_y, \quad w = \eta_y, \quad v = -(\xi_x + \eta_z) \quad (1.4)$$

for some functions ξ and η in t, x, y, z . The equations (1.3)-(1.6) have the similar symmetry transformation as that in (1.9). Modulo this transformation, we assume

$$\xi = f(t, x, z) + \kappa(t, x, z)y + g(t, x, z)H(t, \sigma(t, x, z)y), \quad (1.14)$$

and

$$\eta = \tau(t, x, z)y + \zeta(t, x, z)\Phi(t, \sigma(t, x, z)y), \quad (1.15)$$

where $f, g, \kappa, \sigma, \tau, \zeta$ are three-variable functions to be determined, and $H(t, \varpi)$ and $\Phi(t, \varpi)$ are more sophisticated two-variable functions that are either first given or are to be solved under certain conditions. Again the terms $\kappa(t, x, z)y$ and $\tau(t, x, z)y$ are used to make H and Φ more flexible. With the ansatz (1.14) and (1.15), the equations (1.3) and (1.4) are reduced to a system of several more complicated nonlinear partial differential equations in t, x, z . This again in a way shows that our method is much more powerful than the Lie group method. One of the main technique of solving the system of nonlinear partial differential equations in t, x, z is to use time-dependent orthogonal transformations (moving frames). Imposing some technical conditions on the above undetermined functions, we are able to solve the system of partial differential equations. In this way, we obtain ten families of function-parameter exact solutions of the three-dimensional classical non-steady boundary layer problem (1.3)-(1.5). For instance, we have the following solutions: (1) for any function ι of z , any function α of t and arbitrary functions \mathfrak{S}, φ in $\hat{\varpi} = \alpha - \iota$:

$$u = \frac{\alpha''}{2\alpha'}x - \frac{\varphi}{\sqrt{\alpha'}} \left[\exp \left(-2\nu\mathfrak{S}^2 \int \frac{dz}{(\iota')^2} \right) \right] \cos \left(\frac{\sqrt{\alpha'}\mathfrak{S}y}{\iota'} \right), \quad (1.16)$$

$$v = \frac{\alpha'\iota''y}{(\iota')^2} - \frac{\alpha''}{2\alpha'}y, \quad w = \frac{\alpha'}{\iota'}, \quad (6.17)$$

$$p = -\frac{\rho}{2} \left(\frac{2\alpha'\alpha'' - (\alpha'')^2}{4(\alpha')^2}x^2 + \frac{(\alpha')^2}{(\iota')^2} + \alpha'' \int \frac{dz}{\iota'} \right) \quad (6.18)$$

(see Theorem 6.1); (2) in terms of the moving frame

$$\tilde{\varpi} = x \cos \gamma + z \sin \gamma, \quad \hat{\varpi} = z \cos \gamma - x \sin \gamma, \quad (1.19)$$

$$\begin{aligned} u = & \frac{\beta^2\gamma' + 2\nu(\beta'\cos\gamma - 2\beta\gamma'\sin\gamma - 6\gamma')}{4\nu(\beta - 6\nu\sin\gamma)}(\tilde{\varpi} + 2\hat{\varpi}\tan\gamma) \\ & - \left(\frac{1}{6\nu}\beta\gamma'\tan\gamma + \gamma'\sec\gamma \right) \hat{\varpi} - (6\nu x + \beta\hat{\varpi})y^{-2}, \end{aligned} \quad (1.20)$$

$$v = \frac{\beta^2\gamma' + 2\nu(\beta' \cos \gamma - 2\beta\gamma' \sin \gamma - 6\gamma')}{4\nu \cos \gamma (\beta - 6\nu \sin \gamma)} y - \frac{\beta\gamma'}{6\nu} y \sec \gamma - 6\nu y^{-1}, \quad (1.21)$$

$$\begin{aligned} w = & \frac{\beta^2\gamma' + 2\nu(\beta' \cos \gamma - 2\beta\gamma' \sin \gamma - 6\gamma')}{4\nu(\beta - 6\nu \sin \gamma)} (\tilde{\omega} \tan \gamma - 2\hat{\omega}) \\ & + \frac{1}{6\nu} \beta\gamma' \hat{\omega} - \gamma' \tilde{\omega} \sec \gamma - (6\nu x + \beta\hat{\omega}) y^{-2} \tan \gamma \end{aligned} \quad (1.22)$$

and p is more complicated and given in (7.26), where β and γ are arbitrary functions of t (see Theorem 7.1). Applying the symmetry transformations of the equations (1.3)-(1.5) (see next section) such as that similar to the one in (1.9), we can get solutions with more parameter functions. For instance, the above second solution would yield to a solution singular on any moving hypersurface, which may be used to study turbulence. In addition, we present another two anzatzes and obtain another two families of function-parameter exact solutions of the three-dimensional problem that are rational functions in y .

Of course, one can view the two-dimensional problem (1.1) and (1.2) as a special case of the three-dimensional problem (1.3)-(1.5). Our results on the two-dimensional problem in Section 3 can be directly applied to the three dimensional problem by assuming that ξ is independent of z and η is independent of x . Applying constant orthogonal transformations (see Section 2), we can obtain more sophisticated solutions of the three-dimensional problem. So obtained solutions of the three-dimensional problem will not be given. Thus not all the solutions of the two-dimensional problem in Section 3 are the specializations of the solutions of the three-dimensional problem in later sections. Our another purpose of presenting the solutions of the two-dimensional problem in Section 3 is to let the reader first understand our techniques in simpler version and then to better understand much more sophisticated techniques of solving the three-dimensional problem in later sections. Indeed, due to the technicality, the author himself keeps referring the results and arguments on the two-dimensional problem in Section 3 when he solves the three-dimensional problem in later sections.

For convenience, we always assume that all the involved partial derivatives of related functions always exist and we can change orders of taking partial derivatives. As we all know that in general, it is impossible to find all the solutions of nonlinear partial differential equations both analytically and algebraically. In our arguments throughout this paper, we always search for reasonable sufficient conditions of obtaining exact solutions. For instance, we treat nonzero functions as “nonzero constants” for this purpose because our approaches in this paper are completely algebraic. Of course, one can used our methods in this paper to get more solutions, in particular, by considering the support and

discontinuity of the related functions.

The paper is organized as follows. In Section 2, we present the known Lie point symmetries of the equations (1.1) and (1.2) due to Ovsiannikov [O], and the Lie point symmetries of the equations (1.3)-(1.5) due to Garaev [G] and by Vereschagina [V]. In later sections, we always try to exclude the obvious ingredients in our solutions induced by these symmetry transformations. In other words, we search for the solutions modulo these transformations. We solve the two-dimensional classical non-steady boundary layer equations (1.1) and (1.2) in Section 3. Starting from Section 4, we solve the three-dimensional classical non-steady boundary layer equations (1.3)-(1.5) according to the types of the involved functions with respect to the variable y . First in Section 4, we find solutions of the equations (1.3)-(1.5) containing a single exponential function in y . In Section 5, we look for solutions of the equations (1.3)-(1.5) with distinct exponential functions in y . In Section 6, we obtain solutions of the equations (1.3)-(1.5) with trigonometric and hyperbolic-type functions in y . Finally in Section 7, we present solutions the equations (1.3)-(1.5) which are rational functions with respect to the variable y .

2 Lie Point Symmetries

In this section, we will present all the known Lie point symmetries of concerned boundary layer problems.

Denote by $\mathcal{F}_{(i)}(x_1, \dots, x_n)$ the space of functions in x_1, \dots, x_n , whose all partial derivatives up i th order exists. Ovsiannikov [O] found the following symmetric group G_2 generated by the transformations

$$\{T_{1a}, T_{2b}, T_{3b}, T_{4\alpha}, T_{5\beta}, T_{6\sigma} \mid a, b \in \mathbb{R}, b \neq 0; \alpha \in \mathcal{F}_{(2)}(t); \beta \in \mathcal{F}_{(0)}(t); \sigma \in \mathcal{F}_{(1)}(t, x)\} \quad (2.1)$$

with

$$T_{1a}(u) = u(t + at, x, y), \quad T_{1a}(v) = v(t + at, x, y), \quad T_{1a}(p) = p(t + at, x); \quad (2.2)$$

$$T_{2b}(u) = u(b^2t, b^2x, by), \quad T_{2b}(v) = b^{-1}v(b^2t, b^2x, by), \quad T_{2b}(p) = p(b^2t, b^2x); \quad (2.3)$$

$$T_{3b}(u) = bu(t, bx, by), \quad T_{3b}(v) = v(t, bx, by), \quad T_{3b}(p) = b^2p(t, bx); \quad (2.4)$$

$$T_{4\alpha}(u) = u(t, x + \alpha, y) - \alpha', \quad T_{4\alpha}(v) = v(t, x + \alpha, y), \quad T_{4\alpha}(p) = p(t, x + \alpha) + \alpha''x; \quad (2.5)$$

$$T_{5\beta}(u) = u, \quad T_{5\beta}(v) = v, \quad T_{5\beta}(p) = p + \beta; \quad (2.6)$$

$$T_{6\sigma}(u) = u(t, x, y + \sigma), \quad T_{6\sigma}(v) = v(t, x, y + \sigma) - \sigma_t - \sigma_x u, \quad T_{6\sigma}(p) = p. \quad (2.7)$$

The elements of G_2 transform the solutions of the two-dimensional classical non-steady boundary layer equations (1.1) and (1.2) into solutions. For instance, the transformation $T_{4\alpha}T_{5\gamma}$ transforms a solution with zero pressure into a solution with the pressure $p = \alpha''x + \gamma$ and the transformation $T_{6\sigma}$ may change a solution to the one with an additional two-variable parameter function. In this way, we can always transform our solutions to the ones with nonzero pressure and at least one two-variable parameter function. For simplicity, we only write our solutions of (1.1) and (1.2) as simple as possible modulo the group G_2 . We will do the same thing for the solutions of the three dimensional problems (1.3)-(1.5).

The symmetry group G_3 of the equations (1.3)-(1.5) are generated by the transformations

$$\begin{aligned} &\{T_{1a}, T_{2a}, T_{3b}, T_{4b}, T_{5\alpha}, T_{6\alpha}, T_{7\beta}, T_{8\sigma} \mid a, b \in \mathbb{R}, b \neq 0; \\ &\alpha \in \mathcal{F}_{(2)}(t); \beta \in \mathcal{F}_{(0)}(t); \sigma \in \mathcal{F}_{(1)}(t, x, z)\} \end{aligned} \quad (2.8)$$

with

$$T_{1a}(\psi) = \psi(t + a, x, z, y), \quad \psi = u, v, w, p; \quad (2.9)$$

$$\begin{aligned} T_{2a}(u) &= u(t, x \cos a + z \sin a, -x \sin a + z \cos a, y) \cos a \\ &\quad + w(t, x \cos a + z \sin a, -x \sin a + z \cos a, y) \sin a, \end{aligned} \quad (2.10)$$

$$\begin{aligned} T_{2a}(w) &= -u(t, x \cos a + z \sin a, -x \sin a + z \cos a, y) \sin a \\ &\quad + w(t, x \cos a + z \sin a, -x \sin a + z \cos a, y) \cos a, \end{aligned} \quad (2.11)$$

$$T_{2a}(\psi) = \psi(t, x \cos a + z \sin a, -x \sin a + z \cos a, y), \quad \psi = v, p; \quad (2.12)$$

$$T_{3b}(u) = bu(t, bx, y, bz), \quad T_{3b}(v) = v(t, bx, y, bz), \quad (2.13)$$

$$T_{3b}(w) = bw(t, bx, y, bz), \quad T_{3b}(p) = b^2p(t, bx, y, bz); \quad (2.14)$$

$$T_{4b}(u) = b^{-2}u(b^2t, x, by, z), \quad T_{4b}(v) = b^{-1}v(b^2t, x, by, z), \quad (2.15)$$

$$T_{4b}(w) = b^{-2}w(b^2t, x, by, z), \quad T_{4b}(p) = b^{-4}p(b^2t, x, by, z); \quad (2.16)$$

$$T_{5\alpha}(u) = u(t, x + \alpha, y, z) - \alpha', \quad T_{5\alpha}(v) = v(t, x + \alpha, y, z), \quad (2.17)$$

$$T_{5\alpha}(w) = w(t, x + \alpha, y, z), \quad T_{5\alpha}(p) = p(t, x + \alpha, z) + \rho\alpha''x; \quad (2.18)$$

$$T_{6\alpha}(w) = w(t, x, y, z + \alpha) - \alpha', \quad T_{6\alpha}(v) = v(t, x, y, z + \alpha), \quad (2.19)$$

$$T_{6\alpha}(u) = u(t, x, y, z + \alpha), \quad T_{6\alpha}(p) = p(t, x, z + \alpha) + \rho\alpha''z; \quad (2.20)$$

$$T_{7\beta}(u) = u, \quad T_{7\beta}(v) = v, \quad T_{7\beta}(w) = w, \quad T_{7\beta}(p) = p + \beta; \quad (2.21)$$

$$T_{8\sigma}(u) = u(t, x, y + \sigma, z), \quad T_{8\sigma}(v) = v(t, x, y + \sigma) - \sigma_t - \sigma_x u - \sigma_z v, \quad (2.22)$$

$$T_{8\sigma}(u) = w(t, x, y + \sigma, z), \quad T_{8\sigma}(p) = p. \quad (2.23)$$

3 Solutions for the 2-D Problem

In this section, we will find various function-parameter exact solutions for the two-dimensional classical non-steady boundary layer equations (1.1) and (1.2).

Solving the second equation in (1.2), we get

$$u = \xi_y, \quad v = -\xi_x \quad (3.1)$$

for some function ξ in t, x, y . Now the equation (1.1) becomes

$$\xi_{yt} + \xi_y \xi_{xy} - \xi_x \xi_{yy} + p_x = \xi_{yyy}. \quad (3.2)$$

Assume that ξ is of the form:

$$\xi = f(t, x) + \kappa(t, x)y + g(t, x)H(t, y), \quad (3.3)$$

where $\kappa(t, x)$, $g(t, x)$, $f(t, x)$ and $H(t, y)$ are two-variable functions. Then

$$\xi_y = \kappa + gH_y, \quad \xi_x = f_x + \kappa_x y + g_x H, \quad \xi_{yt} = \kappa_t + g_t H_y + gH_{yt}, \quad (3.4)$$

$$\xi_{yx} = \kappa_x + g_x H_y, \quad \xi_{yy} = gH_{yy}, \quad \xi_{yyy} = gH_{yyy}. \quad (3.5)$$

In particular, the nonlinear term in (3.2) becomes

$$\begin{aligned} & \xi_y \xi_{xy} - \xi_x \xi_{yy} = (\kappa + gH_y)(\kappa_x + g_x H_y) - (f_x + \kappa_x y + g_x H)gH_{yy} \\ &= \kappa\kappa_x + (\kappa g_x + \kappa_x g)H_y - (f_x + \kappa_x y)gH_{yy} + gg_x[H_y^2 - HH_{yy}]. \end{aligned} \quad (3.6)$$

In order to linearize (3.6) with respect to H , we divided it into the following cases.

First we consider $H = e^{\gamma y}$. Then

$$\xi_{yt} = \kappa_t + (\gamma g_t + \gamma' g)e^{\gamma y} + \gamma \gamma' g y e^{\gamma y}. \quad (3.7)$$

So (3.2) becomes

$$\kappa_t + \kappa\kappa_x + [\gamma' g + \gamma(g_t + \gamma' g y + \kappa g_x + \kappa_x g - \gamma f_x - \gamma \kappa_x g y - \gamma^2 g)]e^{\gamma y} + p_x = 0, \quad (3.8)$$

which is implied by the following system of partial differential equations:

$$\kappa_t + \kappa\kappa_x + p_x = 0, \quad \gamma' - \gamma\kappa_x = 0, \quad (3.9)$$

$$\gamma'g + \gamma g_t + \kappa\gamma g_x + \gamma\kappa_x g - \gamma^3 g - \gamma^2 g f_x = 0. \quad (3.10)$$

So

$$\kappa = \frac{\gamma'}{\gamma}x + \gamma_0 \quad (3.11)$$

for some function γ_0 of t . Moreover, (3.10) implies

$$f_x = \frac{2\gamma'}{\gamma^2} + \frac{g_t + \gamma_0 g_x}{\gamma g} + \frac{\gamma' x g_x}{\gamma^2 g} - \gamma. \quad (3.12)$$

So

$$\xi = f + \frac{\gamma'}{\gamma}xy + \gamma_0 y + g e^{\gamma y}. \quad (3.13)$$

Hence,

$$p = -\frac{\gamma''}{2\gamma}x^2 - \frac{\gamma'_0\gamma + \gamma_0\gamma'}{\gamma}x \quad (3.14)$$

modulo some $T_{5\beta}$ in (2.6).

Next we consider the case $f = g_x = 0$. Replacing H by gH , we can assume $g = 1$. Now (3.2) becomes the following linear partial differential equation:

$$\kappa_t + \kappa\kappa_x + H_{yt} + \kappa_x H_y - \kappa_x y H_{yy} + p_x = H_{yyy}. \quad (3.15)$$

To simplify the problem, we assume

$$\kappa_t + \kappa\kappa_x + p_x = 0, \quad \kappa_x = \frac{\gamma''}{2\gamma'} \quad (3.16)$$

for some nonzero function γ' of t . Then

$$\kappa = \frac{\gamma''}{2\gamma'}x + \gamma_0 \quad (3.17)$$

for some function γ_0 of t and

$$p = \frac{2\gamma'\gamma'' - (\gamma'')^2}{8(\gamma')^2}x^2 + \frac{\gamma_0\gamma'' + \gamma'_0\gamma'}{2\gamma'}x \quad (3.18)$$

modulo the transformation in (2.6). Now (3.15) becomes

$$H_{yt} + \frac{\gamma''}{2\gamma'}(H_y - yH_{yy}) = H_{yyy}. \quad (3.19)$$

Write

$$H_y = \frac{1}{\sqrt{\gamma'}}\hat{H}(t, \sqrt{\gamma'}y) \quad (3.20)$$

for some two variable function $\hat{H}(t, \varpi)$ with $\varpi = \sqrt{\gamma'}y$. Then (3.19) turns out to be

$$\hat{H}_t = \gamma'\hat{H}_{\varpi\varpi}. \quad (3.21)$$

One can use Fourier expansion to solve the above equation with given initial condition. In particular, discontinuous solutions can be found in this way. For the algebraic neatness, we just want to find globally analytic solution with respect to the spacial variables x and y . So we take the following solution of (3.21):

$$\hat{H} = \sum_{i=1}^m d_i e^{(a_i^2 - b_i^2)\gamma + a_i \varpi} \sin(b_i(2a_i\gamma + \varpi) + c_i), \quad (3.22)$$

where a_i, b_i, c_i, d_i are real numbers. Hence, we have

$$H_y = \frac{1}{\sqrt{\gamma'}} \sum_{i=1}^m d_i e^{(a_i^2 - b_i^2)\gamma + a_i \sqrt{\gamma'} y} \sin(b_i(2a_i\gamma + \sqrt{\gamma'} y) + c_i). \quad (3.23)$$

By (3.1), we have the following result, where the first solution follows from a direct observation the equations in (1.1) and (1.2).

Theorem 3.1. *We have the following solutions of the two-dimensional classical non-steady boundary layer problem (1.1) and (1.2): (1)*

$$u = y, \quad v = -\kappa_x, \quad p = \kappa, \quad (3.24)$$

where κ is an arbitrary function of t, x ; (2)

$$u = \frac{\gamma' x}{\gamma} + \gamma_0 + \gamma g e^{\gamma y}, \quad (3.25)$$

$$v = \gamma - \frac{2\gamma'}{\gamma^2} - \frac{g_t + \gamma_0 g_x}{\gamma g} - \frac{\gamma' x g_x}{\gamma^2 g} - \frac{\gamma' y}{\gamma} - g_x e^{\gamma y}. \quad (3.26)$$

and p is given in (3.14), where γ, γ_0 are any nonzero functions of t and g is any nonzero function of t, x ; (3)

$$u = \frac{\gamma'' x}{2\gamma'} + \gamma_0 + \frac{1}{\sqrt{\gamma'}} \sum_{i=1}^m d_i e^{(a_i^2 - b_i^2)\gamma + a_i \sqrt{\gamma'} y} \sin(b_i(2a_i\gamma + \sqrt{\gamma'} y) + c_i), \quad (3.27)$$

$$v = -\frac{\gamma'' y}{2\gamma'} \quad (3.28)$$

and p is given in (3.18), where γ, γ_0 are any functions of t , and a_i, b_i, c_i, d_i are real constants.

Next we consider the case $f = \partial_y(H_y^2 - HH_{yy}) = 0$. Again let γ be a function of t . For $a \in \mathbb{R}$, we denote

$$\vartheta_0 = \frac{e^{\gamma y} - a e^{-\gamma y}}{2}, \quad \vartheta_1 = \sin(\gamma y), \quad (3.29)$$

$$\hat{\vartheta}_0 = \frac{e^{\gamma y} + a e^{-\gamma y}}{2}, \quad \hat{\vartheta}_1 = \cos(\gamma y). \quad (3.30)$$

Given $r \in \{0, 1\}$. Assume

$$H = \vartheta_r. \quad (3.31)$$

Then

$$H_y^2 - HH_{yy} = (\delta_{0,r}a + \delta_{1,r})\gamma^2. \quad (3.32)$$

By (3.6),

$$\xi_y \xi_{yx} - \xi_x \xi_{yy} = \kappa \kappa_x + (\kappa g_x + \kappa_x g) \gamma \hat{\vartheta}_r - (-1)^r \gamma^2 (f_x + \kappa_x y) g \vartheta_r + (\delta_{0,r}a + \delta_{1,r}) \gamma^2 g g_x. \quad (3.33)$$

Since

$$\xi_{yt} = \kappa_t + (g_t \gamma + g \gamma') \hat{\vartheta}_r - (-1)^r \gamma' \gamma g y \vartheta_r, \quad \xi_{yyy} = (-1)^r \gamma^3 \hat{\vartheta}_r, \quad (3.34)$$

$$\begin{aligned} & \kappa_t + \kappa \kappa_x + p_x + [\gamma' g + \gamma (g_t + \kappa g_x + \kappa_x g - (-1)^r \gamma^2 g)] \hat{\vartheta}_r \\ & - (-1)^r \gamma (\gamma' + \gamma \kappa_x) y g \vartheta_r + (\delta_{0,r}a + \delta_{1,r}) \gamma^2 g g_x = 0, \end{aligned} \quad (3.35)$$

which is implied by the following equations:

$$\kappa_t + \kappa \kappa_x + (\delta_{0,r}a + \delta_{1,r}) \gamma^2 g g_x + p_x = 0, \quad \gamma \kappa_x + \gamma' = 0, \quad (3.36)$$

$$g \gamma' + (g_t + \kappa g_x + \kappa_x g - (-1)^r \gamma^2 g) \gamma = 0. \quad (3.37)$$

For convenience of solving the above equations, we assume

$$\gamma = \sqrt{\beta'} \implies \frac{\gamma'}{\gamma} = \frac{\beta''}{2\beta'}. \quad (3.38)$$

So we take

$$\kappa = \frac{\beta' x}{2\beta'}, \quad g = \frac{e^{(-1)^r \beta}}{\beta'} \mathfrak{S} \left(\frac{x}{\sqrt{\beta'}} \right) \quad (3.39)$$

for some one-variable function \mathfrak{S} . Moreover,

$$p = \frac{(\beta'')^2 - 2\beta' \beta'''}{4(\beta')^2} x^2 - \frac{\delta_{0,r}a + \delta_{1,r}}{2\beta'} e^{(-1)^r 2\beta} \left[\mathfrak{S} \left(\frac{x}{\sqrt{\beta'}} \right) \right]^2 \quad (3.40)$$

modulo the transformation in (2.6) and

$$\xi = \frac{\beta' xy}{2\beta'} + \frac{e^{(-1)^r \beta}}{\beta'} \mathfrak{S} \left(\frac{x}{\sqrt{\beta'}} \right) \vartheta_r. \quad (3.41)$$

According to (3.1), we have:

Theorem 3.2. *Let β be any nonzero increasing function of t and let \mathfrak{S} be any one-variable function. We have the following solutions of the two-dimensional classical non-steady boundary layer problem (1.1) and (1.2): (1)*

$$u = \frac{\beta' x}{2\beta'} + \frac{e^\beta}{2\sqrt{\beta'}} \mathfrak{S} \left(\frac{x}{\sqrt{\beta'}} \right) (e^{\sqrt{\beta'} y} + a e^{-\sqrt{\beta'} y}), \quad (3.42)$$

$$v = -\frac{\beta''y}{2\beta'} - \frac{e^\beta}{2\sqrt{(\beta')^3}} \mathfrak{S}'\left(\frac{x}{\sqrt{\beta'}}\right) (e^{\sqrt{\beta'}y} - ae^{-\sqrt{\beta'}y}) \quad (3.43)$$

and p is given in (3.40) with $r = 0$, where $a \in \mathbb{R}$; (2)

$$u = \frac{\beta''x}{2\beta'} + \frac{e^{-\beta}}{\sqrt{\beta'}} \mathfrak{S}\left(\frac{x}{\sqrt{\beta'}}\right) \cos(\sqrt{\beta'}y), \quad (3.44)$$

$$v = -\frac{\beta''y}{2\beta'} - \frac{e^{-\beta}}{\sqrt{(\beta')^3}} \mathfrak{S}'\left(\frac{x}{\sqrt{\beta'}}\right) \sin(\sqrt{\beta'}y) \quad (3.45)$$

and p is given in (3.40) with $r = 1$.

Finally, we look for ξ of the following form:

$$\xi = \kappa(t, x)y + g(t, x) + h(t, x)y^{-1}, \quad (3.46)$$

where κ, g, h are functions of t, x with $h \not\equiv 0$. Then

$$\xi_y = \kappa - hy^{-2}, \quad \xi_{yt} = \kappa_t - h_t y^{-2}, \quad \xi_{yx} = \kappa_x - h_x y^{-2}, \quad (3.47)$$

$$\xi_x = \kappa_x y + g_x + h_x y^{-1}, \quad \xi_{yy} = 2hy^{-3}, \quad \xi_{yyy} = -6hy^{-4}. \quad (3.48)$$

So (3.2) becomes

$$\kappa_t - h_t y^{-2} + (\kappa - hy^{-2})(\kappa_x - h_x y^{-2}) - 2hy^{-3}(\kappa_x y + g_x + h_x y^{-1}) + p_x = -6hy^{-4}, \quad (3.49)$$

equivalently,

$$\kappa_t + \kappa\kappa_x + p_x - (h_t + \kappa h_x + 3\kappa_x h)y^{-2} - 2hg_x y^{-3} - hh_x y^{-4} = -6hy^{-4}. \quad (3.50)$$

Moreover, (3.50) is implied by the following equations:

$$\kappa_t + \kappa\kappa_x + p_x = 0, \quad h_t + \kappa h_x + 3\kappa_x h = 0, \quad (3.51)$$

$$g = 0, \quad h_x = 6. \quad (3.52)$$

So $h = 6x$ modulo $T_{4\alpha}$ in (2.5).

According to the second equation in (3.51),

$$6\kappa + 18x\kappa_x = 0 \implies \kappa = \gamma x^{-1/3} \quad (3.53)$$

for some function γ of t . Thus

$$\xi = \gamma x^{-1/3}y + 6xy^{-1}. \quad (3.54)$$

Moreover, $\kappa_t = \gamma' x^{-1/3}$. Thus

$$p_x = \frac{\gamma^2}{3} x^{-5/3} - \gamma' x^{-1/3} \implies p = \frac{3\gamma'}{2} x^{2/3} - \frac{\gamma^2}{2} x^{-2/3} \quad (3.55)$$

by modulo $T_{5\beta}$ in (2.6) by the first equation in (3.51). Therefore, (3.1) implies:

Theorem 3.3. *Let γ be any function of t . We have the following solutions of the two-dimensional classical non-steady boundary layer problem (1.1) and (1.2):*

$$u = \gamma x^{-1/3} - 6xy^{-2}, \quad v = \frac{\gamma}{3} x^{-4/3} - 6y^{-1}, \quad p = \frac{3\gamma'}{2} x^{2/3} - \frac{\gamma^2}{2} x^{-2/3}. \quad (3.56)$$

4 3-D Uniform Exponential Approach

In this section, we will find certain function-parameter exact solutions with single exponential function in y for the three-dimensional non-steady boundary layer equations (1.3)-(1.5).

Solving the second equation in (1.5), we get

$$u = \xi_y, \quad w = \eta_y, \quad v = -(\xi_x + \eta_z) \quad (4.1)$$

for some functions ξ and η in t, x, y, z . Now the equation (1.3) and (1.4) become

$$\xi_{yt} + \xi_y \xi_{yx} - (\xi_x + \eta_z) \xi_{yy} + \eta_y \xi_{yz} + \frac{1}{\rho} p_x = \nu \xi_{yyy}, \quad (4.2)$$

$$\eta_{yt} + \xi_y \eta_{yx} - (\xi_x + \eta_z) \eta_{yy} + \eta_y \eta_{yz} + \frac{1}{\rho} p_z = \nu \eta_{yyy}. \quad (4.3)$$

First we assume that ξ and η are of the form:

$$\xi = f(t, x, z) + \kappa(t, x, z)y + g(t, x, z)H(t, \sigma(t, x, z)y), \quad (4.4)$$

and

$$\eta = \tau(t, x, z)y + \zeta(t, x, z)\Phi(t, \sigma(t, x, z)y), \quad (4.5)$$

where $f, g, \kappa, \sigma, \tau, \zeta$ are three-variable functions, and $H(t, \varpi)$ and $\Phi(t, \varpi)$ are two variable functions, where $\varpi = \sigma y$. Then

$$\xi_y = \kappa + \sigma g H_{\varpi}, \quad \eta_y = \tau + \sigma \zeta \Phi_{\varpi}, \quad (4.6)$$

$$\xi_x = f_x + \kappa_x y + g_x H + \sigma_x g y H_{\varpi}, \quad \eta_z = \tau_z y + \zeta_z \Phi + \sigma_z \zeta \Phi_{\varpi}, \quad (4.7)$$

$$\xi_{yx} = \kappa_x + (\sigma g)_x H_{\varpi} + \sigma \sigma_x g y H_{\varpi\varpi}, \quad \eta_{yx} = \tau_x + (\sigma \zeta)_x \Phi_{\varpi} + \sigma \sigma_x g y \Phi_{\varpi\varpi}, \quad (4.8)$$

$$\xi_{yz} = \kappa_z + (\sigma g)_z H_{\varpi} + \sigma \sigma_z g y H_{\varpi\varpi}, \quad \eta_{yz} = \tau_z + (\sigma \zeta)_z \Phi_{\varpi} + \sigma \sigma_z g y \Phi_{\varpi\varpi}, \quad (4.9)$$

$$\xi_{yy} = \sigma^2 g H_{\varpi\varpi}, \quad \xi_{yyy} = \sigma^3 g H_{\varpi\varpi\varpi}, \quad \eta_{yy} = \sigma^2 \zeta \Phi_{\varpi\varpi}, \quad \eta_{yyy} = \sigma^3 \zeta \Phi_{\varpi\varpi\varpi}. \quad (4.10)$$

Thus the nonlinear term in (4.2) becomes

$$\begin{aligned} & \xi_y \xi_{yx} - (\xi_x + \eta_z) \xi_{yy} + \eta_y \xi_{yz} = (\kappa + \sigma g H_{\varpi})(\kappa_x + (\sigma g)_x H_{\varpi} + \sigma \sigma_x g y H_{\varpi\varpi}) \\ & - \sigma^2 g H_{\varpi\varpi} (f_x + (\kappa_x + \tau_z) y + g_x H + \sigma_x g y H_{\varpi} + \zeta_z \Phi + \sigma_z \zeta y \Phi_{\varpi}) \\ & + (\tau + \sigma \zeta \Phi_{\varpi})(\kappa_z + (\sigma g)_z H_{\varpi} + \sigma \sigma_z g y H_{\varpi\varpi}) \\ = & \kappa \kappa_x + \tau \kappa_z + [(\sigma g \kappa)_x + (\sigma g)_z \tau] H_{\varpi} + \sigma \zeta \kappa_z \Phi_{\varpi} + \sigma g \{[\kappa \sigma_x + \tau \sigma_z - \sigma(\kappa_x + \tau_z)] y \\ & - \sigma f_x\} H_{\varpi\varpi} + \sigma g (\sigma g)_x H_{\varpi}^2 - \sigma^2 g g_x H H_{\varpi\varpi} + \sigma \zeta (\sigma g)_z H_{\varpi} \Phi_{\varpi} - \sigma^2 g \zeta_z H_{\varpi\varpi} \Phi. \end{aligned} \quad (4.11)$$

Symmetrically, the nonlinear term in (4.3) becomes

$$\begin{aligned} & \xi_y \eta_{yx} - (\xi_x + \eta_z) \eta_{yy} + \eta_y \eta_{yz} \\ = & \kappa \tau_x + \tau \tau_z + [(\sigma \zeta \tau)_z + (\sigma \zeta)_x \kappa] \Phi_{\varpi} + \sigma g \tau_x H_{\varpi} + \sigma \zeta \{[\kappa \sigma_x + \tau \sigma_z - \sigma(\kappa_x + \tau_z)] y \\ & - \sigma f_x\} \Phi_{\varpi\varpi} + \sigma \zeta (\sigma \zeta)_z \Phi_{\varpi}^2 - \sigma^2 \zeta \zeta_z \Phi \Phi_{\varpi\varpi} + \sigma g (\sigma \zeta)_x H_{\varpi} \Phi_{\varpi} - \sigma^2 g_x \zeta H \Phi_{\varpi\varpi}. \end{aligned} \quad (4.12)$$

In order to linearize (4.11) and (4.12) with respect to H and Φ , we divide it into ten cases.

Case 1. $H = \Phi = e^{\varpi}$.

In this case,

$$\begin{aligned} & \xi_y \xi_{yx} - (\xi_x + \eta_z) \xi_{yy} + \eta_y \xi_{yz} = \kappa \kappa_x + \tau \kappa_z + \{\sigma g \kappa_x + \sigma \zeta \kappa_z + (\sigma g)_x \kappa + (\sigma g)_z \tau \\ & + \sigma g \{[\kappa \sigma_x + \tau \sigma_z - \sigma(\kappa_x + \tau_z)] y - \sigma f_x\}\} e^{\varpi} + \sigma [\sigma_x g^2 + \zeta (\sigma g)_z - \sigma g \zeta_z] e^{2\varpi}. \end{aligned} \quad (4.13)$$

Moreover,

$$\xi_{yt} = \kappa_t + ((\sigma g)_t + \sigma \sigma_t g y) e^{\varpi}, \quad \xi_{yyy} = \sigma^3 g e^{\varpi}. \quad (4.14)$$

Thus (4.2) becomes

$$\begin{aligned} & \kappa_t + \kappa \kappa_x + \tau \kappa_z + \{(\sigma g)_t + \sigma g \kappa_x + \sigma \zeta \kappa_z + (\sigma g)_x \kappa + (\sigma g)_z \tau + \sigma g \{[\sigma_t + \kappa \sigma_x + \tau \sigma_z \\ & - \sigma(\kappa_x + \tau_z)] y - \sigma f_x - \nu \sigma^2\}\} e^{\varpi} + \sigma [\sigma_x g^2 + \zeta (\sigma g)_z - \sigma g \zeta_z] e^{2\varpi} + \frac{1}{\rho} p_x = 0. \end{aligned} \quad (4.15)$$

By symmetry, (4.2) and (4.3) are implied by the following system of partial differential equations:

$$\kappa_t + \kappa \kappa_x + \tau \kappa_z + \frac{1}{\rho} p_x = 0, \quad \tau_t + \kappa \tau_x + \tau \tau_z + \frac{1}{\rho} p_z = 0, \quad (4.16)$$

$$\sigma_t + \kappa\sigma_x + \tau\sigma_z - \sigma(\kappa_x + \tau_z) = 0, \quad (4.17)$$

$$(\sigma g)_t + \sigma g\kappa_x + \sigma\zeta\kappa_z + (\sigma g)_x\kappa + (\sigma g)_z\tau - \sigma^2 g(f_x + \nu\sigma) = 0, \quad (4.18)$$

$$(\sigma\zeta)_t + \sigma g\tau_x + \sigma\zeta\tau_z + (\sigma\zeta)_x\kappa + (\sigma\zeta)_z\tau - \sigma^2\zeta(f_x + \nu\sigma) = 0, \quad (4.19)$$

$$\sigma_x g^2 + \zeta(\sigma g)_z - \sigma g\zeta_z = 0, \quad \sigma_z \zeta^2 + g(\sigma\zeta)_x - \sigma g_x \zeta = 0. \quad (4.20)$$

Observe that (4.20) yields

$$\sigma_x + \sigma_z \frac{\zeta}{g} = \sigma \left(\frac{\zeta}{g} \right)_z, \quad \sigma_z + \sigma_x \frac{g}{\zeta} = \sigma \left(\frac{g}{\zeta} \right)_x. \quad (4.21)$$

Thus we take

$$\left(\frac{\zeta}{g} \right)_z = \left(\frac{\zeta}{g} \right) \left(\frac{g}{\zeta} \right)_x \implies \left(\frac{\zeta}{g} \right) \left(\frac{\zeta}{g} \right)_z = - \left(\frac{\zeta}{g} \right)_x. \quad (4.22)$$

Modulo $T_{5\alpha}, T_{6\alpha}$ in (2.17)-(2.20), we take the solutions:

$$\frac{\zeta}{g} = \tan \gamma, \quad \frac{z}{x}, \quad (4.23)$$

where γ is a function of t . In the first case,

$$\sigma_x + \sigma_z \tan \gamma = 0, \quad (4.24)$$

and we take

$$\sigma = e^{\Im(z \cos \gamma - x \sin \gamma)} \sec \gamma \quad (4.25)$$

for later convenience. In the second case,

$$x\sigma_x + z\sigma_z = \sigma, \quad (4.26)$$

and we take

$$\sigma = x e^{\Im(x/z) + \gamma_1}, \quad (4.27)$$

where γ_1 is a function of t and \Im is a one-variable function. Furthermore, $\sigma g \times (4.19) - \sigma\zeta \times (4.18)$ gives

$$\begin{aligned} & (\sigma\zeta)_t \sigma g - \sigma\zeta(\sigma g)_t + (\sigma g)^2 \tau_x - (\sigma\zeta)^2 \kappa_z + (\sigma\zeta)(\sigma g)(\tau_z - \kappa_x) \\ & + [(\sigma\zeta)_x \sigma g - \sigma\zeta(\sigma g)_x] \kappa + [(\sigma\zeta)_z \sigma g - \sigma\zeta(\sigma g)_z] \tau = 0. \end{aligned} \quad (4.28)$$

Multiplying (4.28) by $1/(\sigma g)^2$, we obtain

$$\left(\frac{\zeta}{g} \right)_t + \tau_x - \left(\frac{\zeta}{g} \right)^2 \kappa_z + \left(\frac{\zeta}{g} \right) (\tau_z - \kappa_x) + \left(\frac{\zeta}{g} \right)_x \kappa + \left(\frac{\zeta}{g} \right)_z \tau = 0. \quad (4.29)$$

Since $p_{xz} = p_{zx}$ by our convention, (4.16) implies

$$(\kappa_z - \tau_x)_t + (\kappa(\kappa_z - \tau_x))_x + (\tau(\kappa_z - \tau_x))_z = 0. \quad (4.30)$$

Assume $\zeta/g = \tan \gamma$. Denote the moving frame

$$\tilde{\omega} = x \cos \gamma + z \sin \gamma, \quad \hat{\omega} = z \cos \gamma - x \sin \gamma. \quad (4.31)$$

Then

$$\partial_{\tilde{\omega}} = \cos \gamma \partial_x + \sin \gamma \partial_z, \quad \partial_{\hat{\omega}} = -\sin \gamma \partial_x + \cos \gamma \partial_z. \quad (4.32)$$

Observe that (4.29) becomes

$$\frac{\gamma'}{\cos^2 \gamma} + \tau_x - \kappa_z \tan^2 \gamma + (\tau_z - \kappa_x) \tan \gamma = 0, \quad (4.33)$$

equivalently,

$$\gamma' + (\cos \gamma \partial_x + \sin \gamma \partial_z)(\tau \cos \gamma - \kappa \sin \gamma) = 0 \sim \gamma' + \partial_{\tilde{\omega}}(\tau \cos \gamma - \kappa \sin \gamma) = 0, \quad (4.34)$$

and so we take

$$\tau \cos \gamma - \kappa \sin \gamma = -\gamma' \tilde{\omega} \implies \tau = \kappa \tan \gamma - \gamma' \tilde{\omega} \sec \gamma \quad (4.35)$$

for the convenience of solving the problem. Note

$$\partial_t(\tilde{\omega}) = \gamma' \hat{\omega}, \quad \partial_t(\hat{\omega}) = -\gamma' \tilde{\omega}. \quad (4.36)$$

Substituting (4.25) and (4.35) into (4.17), we obtain

$$2\gamma' \sin \gamma - 2\gamma' \cos \gamma \tilde{\omega} \mathfrak{S}' - \partial_{\tilde{\omega}}(\kappa) = 0, \quad (4.37)$$

where \mathfrak{S} is a function of $\hat{\omega}$. Thus we take

$$\kappa = 2\gamma' \tilde{\omega} \sin \gamma - \gamma' \tilde{\omega}^2 \mathfrak{S}' \cos \gamma + \varepsilon(t, \hat{\omega}) \quad (4.38)$$

for some two-variable function $\varepsilon(t, \hat{\omega})$. Moreover,

$$\tau = -\gamma' \tilde{\omega} \frac{\cos 2\gamma}{\cos \gamma} - \gamma' \tilde{\omega}^2 \mathfrak{S}' \sin \gamma + \varepsilon \tan \gamma. \quad (4.39)$$

Note

$$\kappa_z - \tau_x = -\gamma' \tilde{\omega}^2 \mathfrak{S}'' + \varepsilon_{\hat{\omega}} \sec \gamma + \gamma'. \quad (4.40)$$

Moreover,

$$\begin{aligned} (\kappa_z - \tau_x)_t &= -\gamma'' \tilde{\omega}^2 \mathfrak{S}'' - 2(\gamma')^2 \tilde{\omega} \hat{\omega} \mathfrak{S}'' + (\gamma')^2 \tilde{\omega}^3 \mathfrak{S}''' \\ &\quad + \sec \gamma (\varepsilon_{\hat{\omega}t} + \gamma' \tan \gamma \varepsilon_{\hat{\omega}} - \gamma' \tilde{\omega} \varepsilon_{\hat{\omega}\hat{\omega}}) + \gamma'', \end{aligned} \quad (4.41)$$

$$[\kappa(\kappa_z - \tau_x)]_x + [\tau(\kappa_z - \tau_x)]_z = (\gamma')^2 \left[4\mathfrak{S}'\mathfrak{S}'' + \mathfrak{S}''' \right] \tilde{\omega}^3 - 3(\gamma')^2 \tan \gamma \times \mathfrak{S}'' \tilde{\omega}^2 - \gamma' [\sec \gamma (2\varepsilon \mathfrak{S}' + \varepsilon_{\hat{\omega}}) + 2\gamma' \mathfrak{S}]_{\hat{\omega}} \tilde{\omega} + \gamma' \tan \gamma (\sec \gamma \varepsilon_{\hat{\omega}} + \gamma'). \quad (4.42)$$

Since the subcase $\gamma \in \mathbb{R}$ is included in Case 3 modulo the transformation in (2.10)-(2.12), we assume $\gamma' \neq 0$. Thus (4.30) is implied by the following system of partial differential equations:

$$\mathfrak{S}''' + 2\mathfrak{S}'\mathfrak{S}'' = 0, \quad (\gamma'' + 3(\gamma')^2 \tan \gamma) \mathfrak{S}'' = 0, \quad (4.43)$$

$$\gamma'' + (\gamma')^2 \tan \gamma + 2\gamma' \varepsilon_{\hat{\omega}} \sec \gamma \tan \gamma + \varepsilon_{\hat{\omega}t} \sec \gamma = 0, \quad (4.44)$$

$$[(\varepsilon \mathfrak{S}' + \varepsilon_{\hat{\omega}}) \sec \gamma + \gamma' \hat{\omega} \mathfrak{S}']_{\hat{\omega}} = 0. \quad (4.45)$$

By (4.44), we take

$$\varepsilon = \iota(\hat{\omega}) \cos^2 \gamma - \gamma' \hat{\omega} \cos \gamma \quad (4.46)$$

for some function ι of $\hat{\omega}$. Moreover, (4.45) is implied by

$$(\mathfrak{S}'(\hat{\omega}) \iota(\hat{\omega}) + \iota'(\hat{\omega}))_{\hat{\omega}} = 0. \quad (4.47)$$

First,

$$\gamma \text{ is arbitrary and } \iota = c_1 + d_1 e^{-a\hat{\omega}} \quad \text{if } \mathfrak{S} = a\hat{\omega} + b, \quad (4.48)$$

where $a, b, c_1, d_1 \in \mathbb{R}$. Assume $\mathfrak{S}'' \neq 0$, we have

$$\mathfrak{S}'''(\hat{\omega}) + 2\mathfrak{S}'(\hat{\omega})\mathfrak{S}''(\hat{\omega}) = 0, \quad \gamma'' + 3(\gamma')^2 \tan \gamma = 0. \quad (4.49)$$

Thus

$$\mathfrak{S} = \ln(a\hat{\omega} + b), \ln \sin(a\hat{\omega} + b), \ln(b e^{a\hat{\omega}} + b_1 e^{-a\hat{\omega}}) \quad (4.50)$$

with $a, b, b_1 \in \mathbb{R}$ and γ is implicitly given by

$$\frac{1 + \sin \gamma}{1 - \sin \gamma} = c e^{dt - 2 \sec \gamma \tan \gamma}, \quad c, d \in \mathbb{R}. \quad (4.51)$$

Furthermore,

$$\iota = c_1(a\hat{\omega} + b) + \frac{d_1}{a\hat{\omega} + b} \quad \text{if } \mathfrak{S} = \ln(a\hat{\omega} + b), \quad (4.52)$$

$$\iota = c_1 \cot(a\hat{\omega} + b) + d_1 \csc(a\hat{\omega} + b) \quad \text{if } \mathfrak{S} = \ln \sin(a\hat{\omega} + b), \quad (4.53)$$

$$\iota = c_1 \frac{b e^{a\hat{\omega}} - b_1 e^{-a\hat{\omega}}}{b e^{a\hat{\omega}} + b_1 e^{-a\hat{\omega}}} + \frac{d_1}{b e^{a\hat{\omega}} + b_1 e^{-a\hat{\omega}}} \quad \text{if } \mathfrak{S} = \ln(b e^{a\hat{\omega}} + b_1 e^{-a\hat{\omega}}). \quad (4.54)$$

According to (4.35), (4.38) and (4.39),

$$\begin{aligned} & \kappa_t + \kappa \kappa_x + \tau \kappa_z \\ = & \kappa_t + \kappa \partial_{\hat{\omega}}(\kappa) \sec \gamma - \gamma' \tilde{\omega} \kappa_z \sec \gamma \\ = & 2(\gamma'' \sin \gamma + (\gamma')^2 \cos \gamma) \tilde{\omega} + 2(\gamma')^2 \hat{\omega} \sin \gamma + \varepsilon_t - \gamma' \tilde{\omega} \varepsilon_{\hat{\omega}} \end{aligned}$$

$$\begin{aligned}
& +((\gamma')^2 \sin \gamma - \gamma'' \cos \gamma) \tilde{\omega}^2 \mathfrak{S}' - 2(\gamma')^2 \hat{\omega} \tilde{\omega} \mathfrak{S}' \cos \gamma + (\gamma')^2 \tilde{\omega}^3 \mathfrak{S}'' \cos \gamma \\
& + (2\gamma' \tilde{\omega} \sin \gamma - \gamma' \tilde{\omega}^2 \mathfrak{S}' \cos \gamma + \varepsilon)(2\gamma' \tan \gamma - 2\gamma' \tilde{\omega} \mathfrak{S}') \\
& - \gamma' \tilde{\omega} (2\gamma' \sin^2 \gamma \sec \gamma - \gamma' (2\tilde{\omega} \mathfrak{S}' \sin \gamma + \tilde{\omega}^2 \mathfrak{S}'' \cos \gamma) + \varepsilon_{\hat{\omega}}) \\
= & 2(\gamma')^2 \tilde{\omega}^3 (\mathfrak{S}'' + (\mathfrak{S}')^2) \cos \gamma - (3(\gamma')^2 \sin \gamma + \gamma'' \cos \gamma) \tilde{\omega}^2 \mathfrak{S}' + \varepsilon_t + 2\gamma' (\varepsilon \tan \gamma \\
& + \gamma' \hat{\omega} \sin \gamma) + 2(\gamma'' \sin \gamma + (\gamma')^2 \sec \gamma - \gamma' (\gamma' \hat{\omega} \mathfrak{S}' \cos \gamma + \varepsilon_{\hat{\omega}} + \mathfrak{S}' \varepsilon)) \tilde{\omega}, \tag{4.55}
\end{aligned}$$

$$\begin{aligned}
& \tau_t + \kappa \tau_x + \tau \tau_z \\
= & \tau_t + \kappa \partial_{\tilde{\omega}}(\tau) \sec \gamma - \gamma' \tilde{\omega} \tau_z \sec \gamma \\
= & \left(4(\gamma')^2 \sin \gamma - (\gamma')^2 \frac{\cos 2\gamma \sin \gamma}{\cos^2 \gamma} - \gamma'' \frac{\cos 2\gamma}{\cos \gamma} \right) \tilde{\omega} - (\gamma')^2 \frac{\cos 2\gamma}{\cos \gamma} \hat{\omega} - 2(\gamma')^2 \hat{\omega} \tilde{\omega} \mathfrak{S}' \sin \gamma \\
& - (\gamma'' \sin \gamma + (\gamma')^2 \cos \gamma) \tilde{\omega}^2 \mathfrak{S}' + (\gamma')^2 \tilde{\omega}^3 \mathfrak{S}'' \sin \gamma + \gamma' \sec^2 \gamma \varepsilon + (\varepsilon_t - \gamma' \tilde{\omega} \varepsilon_{\hat{\omega}}) \tan \gamma \\
& - \gamma' (2\gamma' \tilde{\omega} \sin \gamma - \gamma' \tilde{\omega}^2 \mathfrak{S}' \cos \gamma + \varepsilon) \left(\frac{\cos 2\gamma}{\cos^2 \gamma} + 2\tilde{\omega} \mathfrak{S}' \tan \gamma \right) + \gamma' \tilde{\omega} [\gamma' \cos 2\gamma \tan \gamma \\
& + \gamma' \tilde{\omega} (2\mathfrak{S}' \sin^2 \gamma + \tilde{\omega} \mathfrak{S}'' \sin \gamma \cos \gamma) - \varepsilon_{\hat{\omega}} \sin \gamma] \sec \gamma \\
= & 2(\gamma')^2 \tilde{\omega}^3 (\mathfrak{S}'' + (\mathfrak{S}')^2) \sin \gamma - (3(\gamma')^2 \sin \gamma + \gamma'' \cos \gamma) \tilde{\omega}^2 \mathfrak{S}' \tan \gamma \\
& + 2\gamma' (\gamma' \sec \gamma - (\gamma' \hat{\omega} \mathfrak{S}' \cos \gamma + \varepsilon_{\hat{\omega}} + \mathfrak{S}' \varepsilon)) \tilde{\omega} \tan \gamma \\
& - \frac{\cos 2\gamma}{\cos \gamma} (\gamma'' \tilde{\omega} + (\gamma')^2 \hat{\omega}) + (\varepsilon_t + 2\varepsilon \tan \gamma) \tan \gamma. \tag{4.56}
\end{aligned}$$

Note

$$\mathfrak{S}'' + (\mathfrak{S}')^2 = \hat{b} = \begin{cases} a^2 & \text{in Cases (4.48) and (4.54),} \\ 0 & \text{in Case (4.52),} \\ -a^2 & \text{in Case (4.53),} \end{cases} \tag{4.57}$$

$$\gamma' \hat{\omega} \mathfrak{S}' \cos \gamma + \varepsilon_{\hat{\omega}} + \mathfrak{S}' \varepsilon = \iota' + \mathfrak{S}' \iota = \hat{c} = \begin{cases} ac_1 & \text{in Cases (4.48) and (4.54),} \\ 2ac_1 & \text{in Cases (4.52),} \\ -ac_1 & \text{in Case (4.53),} \end{cases} \tag{4.58}$$

and

$$\varepsilon_t + 2\varepsilon \tan \gamma = (\gamma')^2 \tilde{\omega} \cos \gamma - ((\gamma')^2 \sin \gamma + \gamma'' \cos \gamma) \hat{\omega} \tag{4.59}$$

by (4.46). Moreover,

$$2\tilde{\omega} \sin \gamma - \hat{\omega} \cos \gamma = 3x \sin \gamma \cos \gamma + z(3 \sin^2 \gamma - 1), \tag{4.60}$$

$$-\frac{\cos 2\gamma}{\cos \gamma} \tilde{\omega} - \hat{\omega} \sin \gamma = x(3 \sin^2 \gamma - 1) + z(1 - 3 \cos^2 \gamma) \tan \gamma, \tag{4.61}$$

$$-\frac{\cos 2\gamma}{\cos \gamma} - \sin \gamma \tan \gamma = -\cos \gamma. \tag{4.62}$$

By (4.16), (4.43) and (4.56)-(4.63), we have:

$$\begin{aligned}
p = & \rho \left\{ \frac{3(\gamma')^2 \tan \gamma + \gamma''}{3} \tilde{\omega}^3 \mathfrak{S}' + \frac{(\gamma')^2 [\hat{\omega}^2 - (\hat{b} \tilde{\omega}^2 + 1) \tilde{\omega}^2]}{2} + \gamma' \tilde{\omega}^2 (\hat{c} - \gamma \sec \gamma) \sec \gamma \right. \\
& \left. + \gamma'' \left(xz(1 - 3 \sin^2 \gamma) - \frac{3x^2 \sin 2\gamma}{4} + \frac{z^2(3 \cos^2 \gamma - 1) \tan \gamma}{2} \right) \right\} \tag{4.63}
\end{aligned}$$

modulo the transformation in (1.26).

Modulo the transformation given in (2.22) and (2.23), we take

$$g = \sigma^{-1} = \cos \gamma e^{-\mathfrak{S}(\hat{\omega})} \quad (4.64)$$

by (4.25). According to (4.18), (4.23) and (4.38), we have

$$f_x = 2\gamma' e^{-\mathfrak{S}}(\sin \gamma - \tilde{\omega} \mathfrak{S}' \cos \gamma) - \nu e^{\mathfrak{S}} \sec \gamma. \quad (4.65)$$

Furthermore,

$$\xi = f + (2\gamma' \tilde{\omega} \sin \gamma - \gamma' \tilde{\omega}^2 \mathfrak{S}' \cos \gamma + \varepsilon)y + \cos \gamma \exp(e^{\mathfrak{S}} y \sec \gamma - \mathfrak{S}), \quad (4.66)$$

$$\eta = \left(\varepsilon \tan \gamma - \gamma' \frac{\cos 2\gamma}{\cos \gamma} \tilde{\omega} - \gamma' \tilde{\omega}^2 \mathfrak{S}' \sin \gamma \right) y + \sin \gamma \exp(\sec \gamma e^{\mathfrak{S}} y - \mathfrak{S}). \quad (4.67)$$

By (4.1), we have the following theorem.

Theorem 4.1. *In terms of the notations in (4.31), we have the following solutions of the three-dimensional classical non-steady boundary layer equations (1.3), (1.4) and (1.5):*

$$u = 2\gamma' \tilde{\omega} \sin \gamma - \gamma' \tilde{\omega}^2 \mathfrak{S}' \cos \gamma + \varepsilon + \exp(e^{\mathfrak{S}} y \sec \gamma), \quad (4.68)$$

$$w = \varepsilon \tan \gamma - \gamma' \frac{\cos 2\gamma}{\cos \gamma} \tilde{\omega} - \gamma' \tilde{\omega}^2 \mathfrak{S}' \sin \gamma + \tan \gamma \exp(e^{\mathfrak{S}} y \sec \gamma), \quad (4.69)$$

$$v = 2\gamma' e^{-\mathfrak{S}}(\cos \gamma \tilde{\omega} \mathfrak{S}' - \sin \gamma) + \nu e^{\mathfrak{S}} \sec \gamma + \gamma'(2\tilde{\omega} \mathfrak{S}' - \tan \gamma)y \quad (4.70)$$

and p is given in (4.63), where (1)(4.48) is taken and ε is given (4.46); (2) γ is implicitly determined by (4.51), (4.52)-(4.54) are taken and ε is given (4.46).

Next we consider the case $\zeta/g = z/x$. Recall that we have already determined σ in (4.27). Now (4.29) becomes

$$x^2 \tau_x - z^2 \kappa_z + xz(\tau_z - \kappa_x) - z\kappa + x\tau = 0. \quad (4.71)$$

equivalently,

$$(x\partial_x + z\partial_z + 1) \left(\tau - \frac{z}{x} \kappa \right) = 0. \quad (4.72)$$

Thus

$$\tau = \frac{z}{x} \kappa + x^{-1} \varepsilon(t, x/z) \quad (4.73)$$

for some two-variable function ε . Denote

$$\hat{\omega} = \frac{x}{z}. \quad (4.74)$$

Substituting (4.27) into (4.17), we get

$$x\gamma'_1 + \kappa(1 + \hat{\omega}\mathfrak{S}') - \tau\hat{\omega}^2\mathfrak{S}' - x(\kappa_x + \tau_z) = 0. \quad (4.75)$$

By (4.73), it changes to

$$x\kappa_x + z\kappa_z = x\gamma'_1 + x^{-1}\hat{\omega}^2(\varepsilon_{\hat{\omega}} - \varepsilon\mathfrak{S}'). \quad (4.76)$$

Thus

$$\kappa = x\gamma'_1 + x^{-1}\hat{\omega}^2(\varepsilon\mathfrak{S}' - \varepsilon_{\hat{\omega}}) + \frac{\hat{\omega}}{\sqrt{1 + \hat{\omega}^2}}\vartheta(t, \hat{\omega}), \quad (4.77)$$

where ϑ is another two-variable function and the last terms is so written for later convenience. Moreover,

$$\tau = z\gamma'_1 + x^{-1}\hat{\omega}(\varepsilon\mathfrak{S}' - \varepsilon_{\hat{\omega}}) + \frac{\vartheta}{\sqrt{1 + \hat{\omega}^2}} + x^{-1}\varepsilon. \quad (4.78)$$

For convenience of solving the problem, we denote

$$\phi(t, \hat{\omega}) = \hat{\omega}^2(\varepsilon\mathfrak{S}' - \varepsilon_{\hat{\omega}}), \quad \psi(t, \hat{\omega}) = \hat{\omega}^{-2}\phi + \hat{\omega}^{-1}\varepsilon. \quad (4.79)$$

So

$$\kappa = x\gamma'_1 + x^{-1}\phi + \frac{\hat{\omega}\vartheta}{\sqrt{1 + \hat{\omega}^2}}, \quad \tau = z\gamma'_1 + z^{-1}\psi + \frac{\vartheta}{\sqrt{1 + \hat{\omega}^2}}. \quad (4.80)$$

Observe that

$$\kappa_z = -x^{-2}\hat{\omega}^2\phi_{\hat{\omega}} - x^{-1}\left(\frac{\hat{\omega}^2(\vartheta + \hat{\omega}\vartheta_{\hat{\omega}})}{\sqrt{1 + \hat{\omega}^2}} - \frac{\hat{\omega}^4\vartheta}{\sqrt{(1 + \hat{\omega}^2)^3}}\right), \quad (4.81)$$

$$\tau_x = x^{-2}\hat{\omega}^2\psi_{\hat{\omega}} + x^{-1}\left(\frac{\hat{\omega}\vartheta_{\hat{\omega}}}{\sqrt{1 + \hat{\omega}^2}} - \frac{\hat{\omega}^2\vartheta}{\sqrt{(1 + \hat{\omega}^2)^3}}\right). \quad (4.82)$$

Thus

$$\tau_x - \kappa_z = x^{-2}\hat{\omega}^2(\psi + \phi)_{\hat{\omega}} + x^{-1}\hat{\omega}\sqrt{1 + \hat{\omega}^2}\vartheta_{\hat{\omega}} \quad (4.83)$$

and

$$(\tau_x - \kappa_z)_t = x^{-2}\hat{\omega}^2(\psi + \phi)_{t\hat{\omega}} + x^{-1}\hat{\omega}\sqrt{1 + \hat{\omega}^2}\vartheta_{t\hat{\omega}}. \quad (4.84)$$

Moreover,

$$\begin{aligned} \kappa(\tau_x - \kappa_z) &= x^{-3}\hat{\omega}^2\phi(\psi + \psi)_{\hat{\omega}} + x^{-2}\left[\frac{\hat{\omega}^3\vartheta}{\sqrt{1 + \hat{\omega}^2}}(\psi + \phi)_{\hat{\omega}} + \hat{\omega}\sqrt{1 + \hat{\omega}^2}\phi\vartheta_{\hat{\omega}}\right] \\ &\quad + x^{-1}\hat{\omega}^2[\gamma'_1(\psi + \phi)_{\hat{\omega}} + \vartheta\vartheta_{\hat{\omega}}] + \gamma'_1\hat{\omega}\sqrt{1 + \hat{\omega}^2}\vartheta_{\hat{\omega}} \end{aligned} \quad (4.85)$$

and

$$\begin{aligned} \tau(\tau_x - \kappa_z) &= x^{-3}\hat{\omega}^3\psi(\psi + \psi)_{\hat{\omega}} + x^{-2}\left[\frac{\hat{\omega}^2\vartheta}{\sqrt{1 + \hat{\omega}^2}}(\psi + \phi)_{\hat{\omega}} + \hat{\omega}^2\sqrt{1 + \hat{\omega}^2}\psi\vartheta_{\hat{\omega}}\right] \\ &\quad + x^{-1}\hat{\omega}[\gamma'_1(\psi + \phi)_{\hat{\omega}} + \vartheta\vartheta_{\hat{\omega}}] + \gamma'_1\sqrt{1 + \hat{\omega}^2}\vartheta_{\hat{\omega}}. \end{aligned} \quad (4.86)$$

Observe that

$$\begin{aligned}
[\kappa(\tau_x - \kappa_z)]_x &= x^{-4} \hat{\omega}^2 (\hat{\omega} \partial_{\hat{\omega}} - 1) (\phi(\psi + \psi)_{\hat{\omega}}) + x^{-1} \gamma'_1 \hat{\omega} (1 + \hat{\omega} \partial_{\hat{\omega}}) (\sqrt{1 + \hat{\omega}^2} \vartheta_{\hat{\omega}}) \\
&+ x^{-3} \left[\hat{\omega}^3 (1 + \hat{\omega} \partial_{\hat{\omega}}) \left(\frac{\vartheta}{\sqrt{1 + \hat{\omega}^2}} (\psi + \phi)_{\hat{\omega}} \right) + \hat{\omega} (\hat{\omega} \partial_{\hat{\omega}} - 1) (\sqrt{1 + \hat{\omega}^2} \phi \vartheta_{\hat{\omega}}) \right] \\
&+ x^{-2} \hat{\omega}^2 (1 + \hat{\omega} \partial_{\hat{\omega}}) [\gamma'_1 (\psi + \phi)_{\hat{\omega}} + \vartheta \vartheta_{\hat{\omega}}]
\end{aligned} \tag{4.87}$$

and

$$\begin{aligned}
[\tau(\tau_x - \kappa_z)]_z &= -x^{-4} \hat{\omega}^4 (\hat{\omega} \partial_{\hat{\omega}} + 3) (\psi(\psi + \psi)_{\hat{\omega}}) - x^{-1} \gamma'_1 \hat{\omega}^2 \partial_{\hat{\omega}} (\sqrt{1 + \hat{\omega}^2} \vartheta_{\hat{\omega}}) \\
&- x^{-3} \hat{\omega}^3 (\hat{\omega} \partial_{\hat{\omega}} + 2) \left[\frac{\vartheta}{\sqrt{1 + \hat{\omega}^2}} (\psi + \phi)_{\hat{\omega}} + \sqrt{1 + \hat{\omega}^2} \psi \vartheta_{\hat{\omega}} \right] \\
&- x^{-2} \hat{\omega}^2 (\hat{\omega} \partial_{\hat{\omega}} + 1) [\gamma'_1 (\psi + \phi)_{\hat{\omega}} + \vartheta \vartheta_{\hat{\omega}}].
\end{aligned} \tag{4.88}$$

By (4.84), (4.87) and (4.88), (4.30) is equivalent to the following system of partial differential equations:

$$(\hat{\omega} \partial_{\hat{\omega}} - 1) (\phi(\psi + \psi)_{\hat{\omega}}) = \hat{\omega}^2 (\hat{\omega} \partial_{\hat{\omega}} + 3) (\psi(\psi + \psi)_{\hat{\omega}}), \tag{4.89}$$

$$(\hat{\omega} \partial_{\hat{\omega}} - 1) (\sqrt{1 + \hat{\omega}^2} \phi \vartheta_{\hat{\omega}}) - \hat{\omega}^2 (\hat{\omega} \partial_{\hat{\omega}} + 2) (\sqrt{1 + \hat{\omega}^2} \psi \vartheta_{\hat{\omega}}) - \frac{\hat{\omega}^2 \vartheta}{\sqrt{1 + \hat{\omega}^2}} (\psi + \phi)_{\hat{\omega}} = 0, \tag{4.90}$$

$$\vartheta_{t\hat{\omega}} + \gamma'_1 \vartheta_{\hat{\omega}} = 0, \quad (\psi + \phi)_{t\hat{\omega}} = 0. \tag{4.91}$$

According to (4.91),

$$\vartheta = \alpha(t) + e^{-\gamma_1} \iota_1(\hat{\omega}), \quad \psi + \phi = \beta_1(t) + \iota(\hat{\omega}), \tag{4.92}$$

where α, β_1, ι_1 and ι are arbitrary one-variable functions. Write (4.89) as

$$\hat{\omega} \partial_{\hat{\omega}} [\iota'(\phi - \hat{\omega}^2 \psi)] = \iota'(\phi + \hat{\omega}^2 \psi). \tag{4.93}$$

Moreover, (4.90) is equivalent to

$$\hat{\omega} (1 + \hat{\omega}^2) \partial_{\hat{\omega}} [\iota'_1(\phi - \hat{\omega}^2 \psi)] - \iota'_1(\phi + \hat{\omega}^4 \psi) - \hat{\omega}^2 (\alpha e^{\gamma_1} + \iota_1) \iota' = 0. \tag{4.94}$$

On the other hand, the first equation in (4.79) yields

$$\varepsilon = e^{\Im(\hat{\omega})} \hat{\varepsilon}(t, \hat{\omega}), \quad \phi = \hat{\omega}^2 e^{\Im(\hat{\omega})} \hat{\varepsilon}_{\hat{\omega}}(t, \hat{\omega}), \tag{4.95}$$

where $\hat{\varepsilon}$ is a function of t and $\hat{\omega}$. According the second equation in (4.79) and (4.92),

$$\phi + \hat{\omega}^{-2} \phi + \hat{\omega}^{-1} \varepsilon = \beta_1 + \iota \implies e^{\Im} [(1 + \hat{\omega}^2) \hat{\varepsilon}_{\hat{\omega}} + \hat{\omega}^{-1} \hat{\varepsilon}] = \beta_1 + \iota. \tag{4.96}$$

Thus

$$\hat{\varepsilon} = \frac{\sqrt{1 + \hat{\omega}^2}}{\hat{\omega}} \left[\beta + \beta_1 \int \frac{\hat{\omega} e^{-\Im}}{\sqrt{(1 + \hat{\omega}^2)^3}} d\hat{\omega} + \int \frac{\hat{\omega} \iota e^{-\Im}}{\sqrt{(1 + \hat{\omega}^2)^3}} d\hat{\omega} \right] \tag{4.97}$$

for some function β of t . Thus

$$\phi = \frac{(\beta_1 + \iota)\hat{\omega}^2}{1 + \hat{\omega}^2} - \frac{e^{\mathfrak{S}}}{\sqrt{1 + \hat{\omega}^2}} \left[\beta + \beta_1 \int \frac{\hat{\omega}e^{-\mathfrak{S}}}{\sqrt{(1 + \hat{\omega}^2)^3}} d\hat{\omega} + \int \frac{\hat{\omega}\iota e^{-\mathfrak{S}}}{\sqrt{(1 + \hat{\omega}^2)^3}} d\hat{\omega} \right]. \quad (4.98)$$

Note

$$\begin{aligned} & \hat{\omega} \partial_{\hat{\omega}} [\iota'(\phi - \hat{\omega}^2 \psi)] = \hat{\omega} \partial_{\hat{\omega}} [\iota'((1 + \hat{\omega}^2)\phi - (\beta_1 + \iota)\hat{\omega}^2)] \\ &= -\hat{\omega} \partial_{\hat{\omega}} \left\{ e^{\mathfrak{S}} \sqrt{1 + \hat{\omega}^2} \iota' \left[\beta + \beta_1 \int \frac{\hat{\omega}e^{-\mathfrak{S}}}{\sqrt{(1 + \hat{\omega}^2)^3}} d\hat{\omega} + \int \frac{\hat{\omega}\iota e^{-\mathfrak{S}}}{\sqrt{(1 + \hat{\omega}^2)^3}} d\hat{\omega} \right] \right\} \\ &= -\hat{\omega} \partial_{\hat{\omega}} [e^{\mathfrak{S}} \sqrt{1 + \hat{\omega}^2} \iota'] \left[\beta + \beta_1 \int \frac{\hat{\omega}e^{-\mathfrak{S}}}{\sqrt{(1 + \hat{\omega}^2)^3}} d\hat{\omega} + \int \frac{\hat{\omega}\iota e^{-\mathfrak{S}}}{\sqrt{(1 + \hat{\omega}^2)^3}} d\hat{\omega} \right] \\ &\quad - \frac{(\beta_1 + \iota)\iota' \hat{\omega}^2}{1 + \hat{\omega}^2}, \end{aligned} \quad (4.99)$$

$$\begin{aligned} \iota'(\phi + \hat{\omega}^2 \psi) &= \iota'((1 - \hat{\omega}^2)\phi + (\beta_1 + \iota)\hat{\omega}^2) = \frac{2(\beta_1 + \iota)\iota' \hat{\omega}^2}{1 + \hat{\omega}^2} \\ &\quad - \frac{\iota'(1 - \hat{\omega}^2)e^{\mathfrak{S}}}{\sqrt{1 + \hat{\omega}^2}} \left[\beta + \beta_1 \int \frac{\hat{\omega}e^{-\mathfrak{S}}}{\sqrt{(1 + \hat{\omega}^2)^3}} d\hat{\omega} + \int \frac{\hat{\omega}\iota e^{-\mathfrak{S}}}{\sqrt{(1 + \hat{\omega}^2)^3}} d\hat{\omega} \right]. \end{aligned} \quad (4.100)$$

Thus (4.93) forces us to take $\iota' = 0$. Replacing β_1 by $\beta_1 + \iota$, we have $\iota = 0$, and (4.93) naturally holds. Similarly, (4.94) yields $\iota_1 = 0$, and (4.94) naturally holds.

According to (4.92), (4.95) and (4.98),

$$\vartheta = \alpha, \quad \phi = \beta_1 - \psi = \frac{\beta_1 \hat{\omega}^2}{1 + \hat{\omega}^2} - \frac{e^{\mathfrak{S}}}{\sqrt{1 + \hat{\omega}^2}} \left(\beta + \beta_1 \int \frac{\hat{\omega}e^{-\mathfrak{S}}}{\sqrt{(1 + \hat{\omega}^2)^3}} d\hat{\omega} \right). \quad (4.101)$$

By (4.80),

$$\kappa = x\gamma_1' + \beta_1 x^{-1} \left(\frac{\hat{\omega}^2}{1 + \hat{\omega}^2} - \frac{e^{\mathfrak{S}}}{\sqrt{1 + \hat{\omega}^2}} \int \frac{\hat{\omega}e^{-\mathfrak{S}}}{\sqrt{(1 + \hat{\omega}^2)^3}} d\hat{\omega} \right) + \frac{\alpha \hat{\omega} - \beta x^{-1} e^{\mathfrak{S}}}{\sqrt{1 + \hat{\omega}^2}}, \quad (4.102)$$

$$\tau = z\gamma_1' + \beta_1 z^{-1} \left(\frac{1}{1 + \hat{\omega}^2} + \frac{e^{\mathfrak{S}}}{\sqrt{1 + \hat{\omega}^2}} \int \frac{\hat{\omega}e^{-\mathfrak{S}}}{\sqrt{(1 + \hat{\omega}^2)^3}} d\hat{\omega} \right) + \frac{\alpha + \beta z^{-1} e^{\mathfrak{S}}}{\sqrt{1 + \hat{\omega}^2}}. \quad (4.103)$$

Modulo the transformation in (2.22) and (2.23), we take $g = \sigma^{-1}$. According to (4.18), (4.27) and (4.101),

$$\begin{aligned} f_x &= \sigma^{-1}(\gamma_1 - x^{-2}\phi) - \nu\sigma = x^{-1}e^{-\mathfrak{S}-\gamma_1} \left(\gamma_1' - \frac{\beta_1}{x^2 + z^2} \right) - \nu x e^{\mathfrak{S}+\gamma_1} \\ &\quad + \frac{x^{-3}e^{-\gamma_1}}{\sqrt{1 + \hat{\omega}^2}} \left(\beta + \beta_1 \int \frac{\hat{\omega}e^{-\mathfrak{S}}}{\sqrt{(1 + \hat{\omega}^2)^3}} d\hat{\omega} \right). \end{aligned} \quad (4.104)$$

According (4.83), $\kappa_z = \tau_x$. Hence (4.16) gives

$$\begin{aligned} p &= \frac{\rho}{2}(\beta'_1 \ln(x^2 + z^2) - \kappa^2 - \tau^2 - \gamma'_1(x^2 + z^2)) - \alpha' \rho \sqrt{x^2 + z^2} \\ &\quad + \rho \int \frac{\hat{\omega}^{-1} e^{\mathfrak{S}}}{\sqrt{1 + \hat{\omega}^2}} \left(\beta' + \beta'_1 \int \frac{\hat{\omega} e^{-\mathfrak{S}}}{\sqrt{(1 + \hat{\omega}^2)^3}} d\hat{\omega} \right) d\hat{\omega}. \end{aligned} \quad (4.105)$$

Recall $\xi = f + \kappa y + g e^{\sigma y}$ and $\eta = \tau y + \zeta e^{\sigma y}$. By (4.1), (4.27), (4.102) and (4.103), we have the following second theorem in this section.

Theorem 4.2. *Let $\hat{\omega} = x/z$. Suppose that $\alpha, \beta, \beta_1, \gamma_1$ are arbitrary functions of t and \mathfrak{S} is any function of $\hat{\omega}$. We have the following solutions of the three-dimensional classical non-steady boundary layer equations (1.3), (1.4) and (1.5):*

$$\begin{aligned} u &= x\gamma'_1 + \beta_1 x^{-1} \left(\frac{\hat{\omega}^2}{1 + \hat{\omega}^2} - \frac{e^{\mathfrak{S}}}{\sqrt{1 + \hat{\omega}^2}} \int \frac{\hat{\omega} e^{-\mathfrak{S}}}{\sqrt{(1 + \hat{\omega}^2)^3}} d\hat{\omega} \right) \\ &\quad + \frac{\alpha \hat{\omega} - \beta x^{-1} e^{\mathfrak{S}}}{\sqrt{1 + \hat{\omega}^2}} + \exp(xy e^{\mathfrak{S} + \gamma_1}), \end{aligned} \quad (4.106)$$

$$\begin{aligned} w &= z\gamma'_1 + \beta_1 z^{-1} \left(\frac{1}{1 + \hat{\omega}^2} + \frac{e^{\mathfrak{S}}}{\sqrt{1 + \hat{\omega}^2}} \int \frac{\hat{\omega} e^{-\mathfrak{S}}}{\sqrt{(1 + \hat{\omega}^2)^3}} d\hat{\omega} \right) \\ &\quad + \frac{\alpha + \beta z^{-1} e^{\mathfrak{S}}}{\sqrt{1 + \hat{\omega}^2}} + \frac{z}{x} \exp(xy e^{\mathfrak{S} + \gamma_1}), \end{aligned} \quad (4.107)$$

$$\begin{aligned} v &= \frac{e^{\mathfrak{S}}}{\sqrt{1 + \hat{\omega}^2}} \left(\frac{(x^2 + z^2) \mathfrak{S}' y}{x z^3} - \frac{y}{x^2} - \frac{e^{-\mathfrak{S} - \gamma_1}}{x^3} \right) \left(\beta + \beta_1 \int \frac{\hat{\omega} e^{-\mathfrak{S}}}{\sqrt{(1 + \hat{\omega}^2)^3}} d\hat{\omega} \right) + \nu x e^{\mathfrak{S} + \gamma_1} \\ &\quad - \frac{\gamma'_1 e^{-\mathfrak{S} - \gamma_1}}{x} + \frac{\beta_1 (xy + e^{-\mathfrak{S} - \gamma_1})}{x(x^2 + z^2)} - \left(2\gamma'_1 + \frac{\alpha}{\sqrt{x^2 + z^2}} \right) y - \frac{y}{x} \exp(xy e^{\mathfrak{S} + \gamma_1}) \end{aligned} \quad (4.108)$$

and p is given in (4.105) with κ in (4.102) and τ in (4.103).

5 3-D Distinct Exponential Approach

In this section, we will find certain function-parameter exact solutions with two distinct exponential functions in y for the three-dimensional non-steady boundary layer equations (1.3)-(1.5).

We use the settings in (4.1)-(4.12) and continue our case-by-case approach.

Case 2. $H = \Phi^{-1} = e^{\varpi}$.

In this case,

$$\begin{aligned} \xi_y \xi_{yx} - (\xi_x + \eta_z) \xi_{yy} + \eta_y \xi_{yz} &= \kappa \kappa_x + \tau \kappa_z - \sigma \zeta (\sigma g)_z - \sigma^2 g \zeta_z - \sigma \zeta \kappa_z e^{-\varpi} + (\sigma g (\sigma g)_x \\ &\quad - \sigma^2 g g_x) e^{2\varpi} + \{(\sigma g \kappa)_x + (\sigma g)_z \tau + \sigma g[(\kappa \sigma_x + \tau \sigma_z - \sigma(\kappa_x + \tau_z))y - \sigma f_x]\} e^{\varpi} \end{aligned} \quad (5.1)$$

by (4.11) and

$$\begin{aligned} &\xi_y \eta_{yx} - (\xi_x + \eta_z) \eta_{yy} + \eta_y \eta_{yz} \\ &= \kappa \tau_x + \tau \tau_z - \sigma g (\sigma \zeta)_x - \sigma^2 g_x \zeta + \sigma g \tau_x e^{\varpi} + (\sigma \zeta (\sigma \zeta)_z - \sigma^2 \zeta \zeta_z) e^{-2\varpi} \\ &\quad + \{-(\sigma \zeta \tau)_z - (\sigma \zeta)_x \kappa + \sigma \zeta[(\kappa \sigma_x + \tau \sigma_z - \sigma(\kappa_x + \tau_z))y - \sigma f_x]\} e^{-\varpi} \end{aligned} \quad (5.2)$$

by (4.12), where we treat $\varpi = \sigma y$. Moreover, (4.14) holds and

$$\eta_{yt} = \tau_t + (\sigma \sigma_t \zeta y - (\sigma \zeta)_t) e^{-\varpi}, \quad \eta_{yyy} = -\sigma^3 \zeta e^{-\varpi}. \quad (5.3)$$

Thus (4.2) and (4.3) are implied by the following system of partial differential equations:

$$\kappa_t + \kappa \kappa_x + \tau \kappa_z - \sigma \zeta (\sigma g)_z - \sigma^2 g \zeta_z + \frac{1}{\rho} p_x = 0, \quad (5.4)$$

$$\tau_t + \kappa \tau_x + \tau \tau_z - \sigma g (\sigma \zeta)_x - \sigma^2 g_x \zeta + \frac{1}{\rho} p_z = 0, \quad (5.5)$$

$$\kappa_z = 0, \quad \tau_x = 0, \quad (5.6)$$

$$\sigma_x = 0, \quad \sigma_z = 0, \quad (5.7)$$

$$\sigma_t + \kappa \sigma_x + \tau \sigma_z - \sigma(\kappa_x + \tau_z) = 0, \quad (5.8)$$

$$(\sigma g)_t + (\sigma g \kappa)_x + (\sigma g)_z \tau - \sigma^2 g(f_x + \nu \sigma) = 0, \quad (5.9)$$

$$(\sigma \zeta)_t + (\sigma \zeta \tau)_z + (\sigma \zeta)_x \kappa + \sigma^2 \zeta(f_x - \nu \sigma) = 0. \quad (5.10)$$

According to (5.7),

$$\sigma = \gamma(t) \quad (5.11)$$

for some nonzero function γ of t . Moreover, (5.8) can be written as

$$\kappa_x + \tau_z = \frac{\gamma'}{\gamma}. \quad (5.12)$$

By (5.6) and (2.17)-(2.20), we take

$$\kappa = \frac{\gamma' - \alpha' \gamma}{\gamma} x, \quad \tau = \alpha' z, \quad (5.13)$$

where α is a function of t . Modulo the transformation in (2.22) and (2.23), we take $g = \gamma^{-1}$, and (5.9) becomes

$$f_x = \frac{\gamma' - \alpha' \gamma}{\gamma^2} - \nu \gamma. \quad (5.14)$$

Substituting (5.11) and (5.14) into (5.10), we get

$$2(\gamma' - \nu\gamma^3)\zeta + \gamma\zeta_t + \alpha'\gamma z\zeta_z + (\gamma' - \alpha'\gamma)x\zeta_x = 0. \quad (5.15)$$

Thus

$$\zeta = \frac{e^{2\nu \int \gamma^2 dt}}{\gamma^2} \phi(e^\alpha x/\gamma, e^{-\alpha} z). \quad (5.16)$$

On the other hand, (5.4)-(5.6), (5.11) and (5.13) say that the compatibility $p_{xz} = p_{zx}$ in (5.4) and (5.5) is implied by

$$(\zeta)_{xx} = (\zeta)_{zz}. \quad (5.17)$$

Let $\epsilon = \pm 1$. The above two equations imply

$$\gamma = \epsilon e^{2\alpha}, \quad \zeta = \exp\left(2\nu \int e^{4\alpha} dt - 4\alpha\right) [\Im(e^{-\alpha}(x+z)) + \iota(e^{-\alpha}(x-z))], \quad (5.18)$$

where \Im and ι are arbitrary one-variable functions. Moreover, (5.4) and (5.5) yield

$$\begin{aligned} p = & \frac{\rho}{2} \left\{ \epsilon \exp\left(2\nu \int e^{4\alpha} dt - 2\alpha\right) [\Im(e^{-\alpha}(x+z)) - \iota(e^{-\alpha}(x-z))] \right. \\ & \left. - (\alpha'' + (\alpha')^2)(x^2 + z^2) \right\} \end{aligned} \quad (5.19)$$

modulo the transformation in (2.21).

By (4.1), we have the following theorem.

Theorem 5.1. *Let \Im, ι be one-variable functions and let α be function of t . Suppose $\epsilon = \pm 1$. We have the following solutions of the three-dimensional classical non-steady boundary layer equations (1.3), (1.4) and (1.5):*

$$u = \alpha'x + \exp(\epsilon e^{2\alpha}y), \quad (5.20)$$

$$w = \alpha'z - \epsilon \exp\left(-\epsilon e^{2\alpha}y + 2\nu \int e^{4\alpha} dt - 2\alpha\right) [\Im(e^{-\alpha}(x+z)) + \iota(e^{-\alpha}(x-z))], \quad (5.21)$$

$$\begin{aligned} v = & \epsilon(\nu e^{2\alpha} - \alpha' e^{2\alpha}) - 2\alpha'y - \exp\left(-\epsilon e^{2\alpha}y + 2\nu \int e^{4\alpha} dt - 5\alpha\right) \\ & \times [\Im'(e^{-\alpha}(x+z)) - \iota'(e^{-\alpha}(x-z))] \end{aligned} \quad (5.22)$$

and p is given in (5.19).

Case 3. $H = e^\varpi$ and $\Phi = 0 = \zeta$.

As in the earlier cases, we take $\sigma g = 1$. Then

$$\begin{aligned} & \xi_y \xi_{yx} - (\xi_x + \eta_z) \xi_{yy} + \eta_y \xi_{yz} = \kappa \kappa_x + \tau \kappa_z + \sigma_x \sigma^{-1} e^{2\varpi} \\ & + \{\kappa_x + [(\kappa \sigma_x + \tau \sigma_z - \sigma(\kappa_x + \tau_z))y - \sigma f_x]\} e^\varpi. \end{aligned} \quad (5.23)$$

by (4.11) and

$$\xi_y \eta_{yx} - (\xi_x + \eta_z) \eta_{yy} + \eta_y \eta_{yz} = \kappa \tau_x + \tau \tau_z + \sigma g \tau_x e^{\varpi}. \quad (5.24)$$

by (4.12). Thus (4.2) and (4.3) are implied by the following system of partial differential equations: (4.16),

$$\sigma_x = 0 = \tau_x, \quad \sigma_t + \tau \sigma_z - \sigma(\kappa_x + \tau_z) = 0, \quad \kappa_x = \sigma(f_x + \nu \sigma). \quad (5.25)$$

We write

$$\sigma = e^{\varsigma(t,z)}, \quad \tau = \varepsilon(t, z) \quad (5.26)$$

for some functions ς and ε of t and z . The second equation in (5.25) yields

$$\kappa_x = \varsigma_t + \varepsilon \varsigma_z - \varepsilon_z = \psi(t, z) \text{ is a function of } t, z \implies \kappa = \phi(t, z) + \psi(t, z)x \quad (5.27)$$

for some function ϕ of t and z . The compatibility $p_{xz} = p_{zx}$ in (4.16) is equivalent to

$$\partial_z(\kappa_t + \kappa \kappa_x + \tau \kappa_z) = 0. \quad (5.28)$$

Note

$$\kappa \kappa_x + \tau \kappa_z = \phi_t + \psi_t x + \psi \phi + \psi^2 x + \varepsilon \phi_z + \varepsilon \psi_z x. \quad (5.29)$$

For simplicity of solving (5.28) for ϕ, ψ and (5.27) for ς , we assume

$$\phi_t + \psi \phi + \varepsilon \phi_z = 0, \quad \psi_t + \psi^2 + \varepsilon \psi_z = 0, \quad \varepsilon = -\frac{\alpha'(t)}{\iota'(z)} \quad (5.30)$$

for some functions α of t and ι of z . Thus

$$\psi = \frac{1}{t + \mathfrak{S}(\alpha + \iota)}, \quad \phi = \frac{\mathfrak{S}_1(\alpha + \iota)}{t + \mathfrak{S}(\alpha + \iota)}, \quad \varsigma = \ln \mathfrak{S}_2(\alpha + \iota)(t + \mathfrak{S}(\alpha + \iota)) - \ln \iota' \quad (5.31)$$

for some one-variable functions $\mathfrak{S}, \mathfrak{S}_1, \mathfrak{S}_2$. Hence

$$\kappa = \frac{x + \mathfrak{S}_1(\alpha + \iota)}{t + \mathfrak{S}(\alpha + \iota)}, \quad \tau = -\frac{\alpha'}{\iota'}, \quad (5.32)$$

$$\sigma = \frac{\mathfrak{S}_2(\alpha + \iota)(t + \mathfrak{S}(\alpha + \iota))}{\iota'}, \quad g = \frac{\iota'}{\mathfrak{S}_2(\alpha + \iota)(t + \mathfrak{S}(\alpha + \iota))}, \quad (5.33)$$

$$p = \frac{\rho}{2} \left(2\alpha'' \int \frac{dz}{\iota'} - \left(\frac{\alpha'}{\iota'} \right)^2 \right) \quad (5.34)$$

modulo the transformation in (2.21). Furthermore, the last equation in (5.25) yields

$$f_x = \frac{\iota'}{\mathfrak{S}_2(\alpha + \iota)(t + \mathfrak{S}(\alpha + \iota))^2} - \frac{\nu}{\iota'} \mathfrak{S}_2(\alpha + \iota)(t + \mathfrak{S}(\alpha + \iota)). \quad (5.35)$$

Recall $\xi = f + \kappa y + g e^{\sigma y}$ and $\eta = \tau y$. By (4.1), we obtain the following theorem:

Theorem 5.2. *Let ι be any function of z and let α be any function of t . Suppose that \Im , \Im_1 and \Im_2 are arbitrary one-variable functions. We have the following solutions of the three-dimensional classical non-steady boundary layer equations (1.3), (1.4) and (1.5):*

$$u = \frac{x + \Im_1(\alpha + \iota)}{t + \Im(\alpha + \iota)} + \exp[y(\iota')^{-1}\Im_2(\alpha + \iota)(t + \Im(\alpha + \iota))], \quad w = -\frac{\alpha'}{\iota'}, \quad (5.36)$$

$$v = \frac{\nu}{\iota'}\Im_2(\alpha + \iota)(t + \Im(\alpha + \iota)) - \frac{\iota'}{\Im_2(\alpha + \iota)(t + \Im(\alpha + \iota))^2} - \frac{y}{t + \Im(\alpha + \iota)} \quad (5.37)$$

and p is given in (5.34).

Case 4. $H = e^\varpi$ and $\Phi = e^{\gamma\varpi}$ for a function γ of t .

Again we assume $\sigma g = 1$. In this case,

$$\begin{aligned} \xi_y \xi_{yx} - (\xi_x + \eta_z) \xi_{yy} + \eta_y \xi_{yz} &= \kappa \kappa_x + \tau \kappa_z + \gamma \sigma \zeta \kappa_z e^{\gamma\varpi} - \sigma \zeta_z e^{(1+\gamma)\varpi} \\ &+ \{\kappa_x + [(\kappa \sigma_x + \tau \sigma_z - \sigma(\kappa_x + \tau_z))y - \sigma f_x]\} e^\varpi + \sigma_x g e^{2\varpi}. \end{aligned} \quad (5.38)$$

by (4.11) and

$$\begin{aligned} \xi_y \eta_{yx} - (\xi_x + \eta_z) \eta_{yy} + \eta_y \eta_{yz} &= \kappa \tau_x + \tau \tau_z + \sigma g \tau_x e^\varpi \\ &+ \gamma \{(\sigma \zeta \tau)_z + (\sigma \zeta)_x \kappa + \gamma \sigma \zeta [(\kappa \sigma_x + \tau \sigma_z - \sigma(\kappa_x + \tau_z))y - \sigma f_x]\} e^{\gamma\varpi} \\ &+ \gamma^2 \sigma \sigma_z \zeta^2 e^{2\gamma\varpi} + \gamma \sigma (g(\sigma \zeta)_x - \gamma \sigma g_x \zeta) e^{(1+\gamma)\varpi}. \end{aligned} \quad (5.39)$$

by (4.12). Moreover, we have (4.14) and

$$\eta_{yt} = \tau_t + ((\gamma \sigma \zeta)_t + \gamma \sigma \sigma_t \zeta y) e^{\gamma\varpi}, \quad \eta_{yyy} = (\gamma \sigma)^3 \zeta e^{\gamma\varpi}. \quad (5.40)$$

Thus (4.2) and (4.3) are implied by the following system of partial differential equations: (4.16),

$$\kappa_z = \zeta_x = \zeta_z = \sigma_z = \sigma_x = \tau_x = 0, \quad (5.41)$$

$$\sigma_t - \sigma(\kappa_x + \tau_z) = 0, \quad \kappa_x = \sigma(f_x + \nu \sigma) \quad (5.42)$$

$$(\gamma \sigma \zeta)_t + \gamma \sigma \zeta \tau_z = \gamma^2 \sigma^2 \zeta (f_x + \nu \gamma \sigma). \quad (5.43)$$

According to (5.41) and (5.42),

$$\sigma = \beta, \quad \kappa = \alpha' x, \quad \tau = \frac{(\beta' - \alpha' \beta)z}{\beta} \quad (5.44)$$

modulo the transformations in (2.17)-(2.20), where α and β are arbitrary function of t . Moreover,

$$f_x = \frac{\alpha'}{\beta} - \nu \beta. \quad (5.45)$$

Then (5.43) becomes

$$(\ln \gamma \beta \zeta)_t + \frac{\beta' - \alpha' \beta}{\beta} = \gamma(\alpha' + \nu(\gamma - 1)\beta^2). \quad (5.46)$$

So

$$\zeta = \frac{1}{\gamma \beta^2} e^{\alpha + \int \gamma(\alpha' + \nu(\gamma - 1)\beta^2) dt}. \quad (5.47)$$

Now

$$p = -\frac{\rho}{2}[(\alpha'' + (\alpha')^2)x^2 + (\beta'' - 2\alpha'\beta' + ((\alpha')^2 - \alpha'')\beta)\beta^{-1}z^2] \quad (5.48)$$

modulo the transformation in (2.21). Moreover,

$$\xi = f + \alpha' xy + \frac{1}{\beta} e^{\beta y}, \quad (5.49)$$

$$\eta = \frac{(\beta' - \alpha'\beta)yz}{\beta} + \frac{1}{\gamma \beta^2} e^{\alpha + \beta \gamma y + \int \gamma(\alpha' + \nu(\gamma - 1)\beta^2) dt}. \quad (5.50)$$

By (4.1), we have the following theorem.

Theorem 5.3. *Let α, β and γ be any functions of t . We have the following solutions of the three-dimensional classical non-steady boundary layer equations (1.3), (1.4) and (1.5):*

$$u = \alpha' x + e^{\beta y}, \quad v = \nu \beta - \frac{\alpha'}{\beta} - \frac{\beta'}{\beta} y, \quad (5.51)$$

$$w = \frac{(\beta' - \alpha'\beta)z}{\beta} + \frac{1}{\beta} e^{\alpha + \beta \gamma y + \int \gamma(\alpha' + \nu(\gamma - 1)\beta^2) dt} \quad (5.52)$$

and p is given in (5.48).

6 3-D Trigonometric and Hyperbolic Approach

In this section, we will find certain function-parameter exact solutions with trigonometric and hyperbolic-type functions in y for the three-dimensional non-steady boundary layer equations (1.3)-(1.5).

We start with the general approach given in (4.1)-(4.12) and continue the case-by-case process. Again we use the notion $\varpi = \sigma(t, x, z)y$. For $a \in \mathbb{R}$, we denote

$$\vartheta_0 = \frac{e^{\varpi} - ae^{-\varpi}}{2}, \quad \hat{\vartheta}_0 = \frac{e^{\varpi} + ae^{-\varpi}}{2}, \quad (6.1)$$

$$\vartheta_1 = \sin \varpi, \quad \hat{\vartheta}_1 = \cos \varpi. \quad (6.2)$$

Case 5. $H = \vartheta_r$ with $r = 0, 1$, and $\zeta = 0 = \Phi$.

Note

$$\begin{aligned} \xi_y \xi_{yx} - (\xi_x + \eta_z) \xi_{yy} + \eta_y \xi_{yz} &= \kappa \kappa_x + \tau \kappa_z + (a\delta_{0,r} + \delta_{1,r}) \sigma^2 g g_x + [(\sigma g \kappa)_x + (\sigma g)_z \tau] \hat{\vartheta}_r \\ -(-1)^r \sigma g \{[\kappa \sigma_x + \tau \sigma_z - \sigma(\kappa_x + \tau_z)]y - \sigma f_x\} \vartheta_r &+ \sigma \sigma_x g^2 \hat{\vartheta}_r^2 \end{aligned} \quad (6.3)$$

by (4.11) and

$$\xi_y \eta_{yx} - (\xi_x + \eta_z) \eta_{yy} + \eta_y \eta_{yz} = \kappa \tau_x + \tau \tau_z + \sigma g \tau_x \hat{\vartheta}_r \quad (6.4)$$

by (4.12). Moreover,

$$\xi_{yt} = \kappa_t + (\sigma g)_t \hat{\vartheta}_r - (-1)^r \sigma \sigma_t g y \vartheta_r, \quad \xi_{yyy} = (-1)^r \sigma^3 g \hat{\vartheta}_r, \quad \eta_{yt} = \tau_t, \quad \eta_{yyy} = 0 \quad (6.5)$$

by (4.4) and (4.5). Thus (4.2) and (4.3) are implied by the following system of partial differential equations:

$$\tau_x = \sigma_x = f_x = 0, \quad (6.6)$$

$$\kappa_t + \kappa \kappa_x + \tau \kappa_z + (a\delta_{0,r} + \delta_{1,r}) \sigma^2 g g_x + \frac{1}{\rho} p_x = 0, \quad \tau_t + \tau \tau_z + \frac{1}{\rho} p_z = 0, \quad (6.7)$$

$$(\sigma g)_t + (\sigma g \kappa)_x + (\sigma g)_z \tau - (-1)^r \nu \sigma^3 g = 0, \quad (6.8)$$

$$\sigma_t + \tau \sigma_z - \sigma(\kappa_x + \tau_z) = 0. \quad (6.9)$$

We take $f = 0$ by (6.6). Let α be a function of t and let ι be a function of z . Denote

$$\hat{\omega} = \alpha - \iota, \quad \tilde{\omega} = \frac{x}{\sqrt{\alpha'}}. \quad (6.10)$$

In order to solve (6.9), we first assume

$$\tau = \frac{\alpha'}{\iota'}, \quad \kappa = \frac{\alpha''}{2\alpha'} x. \quad (6.11)$$

Then

$$\sigma = \frac{\sqrt{\alpha'} \Im(\hat{\omega})}{\iota'} \quad (6.12)$$

for some one-variable function \Im . Multiplying (6.8) by $2\sigma g$, we obtain

$$[(\sigma g)^2]_t + \kappa [(\sigma g)^2]_x + \tau [(\sigma g)^2]_z + 2(\kappa_x - (-1)^r \nu \sigma^2) (\sigma g)^2 = 0. \quad (6.13)$$

So

$$(\sigma g)^2 = \frac{1}{\alpha'} \varsigma(\hat{\omega}, \tilde{\omega}) \exp \left((-1)^r 2\nu \Im^2 \int \frac{dz}{(\iota')^2} \right) \quad (6.14)$$

for some two-variable function ς . By the compatibility $p_{xz} = p_{zx}$, $\partial_x \partial_z (\sigma g)^2 = 0$. Hence we take

$$g = \frac{\iota' \varphi(\hat{\omega})}{\alpha' \Im(\hat{\omega})} \exp \left((-1)^r 2\nu (\Im(\hat{\omega}))^2 \int \frac{dz}{(\iota')^2} \right), \quad (6.13)$$

where φ is a one-variable function. According to (6.5) and (6.9),

$$p = -\frac{\rho}{2} \left(\frac{2\alpha'\alpha'' - (\alpha'')^2}{4(\alpha')^2} x^2 + \frac{(\alpha')^2}{(\iota')^2} + \alpha'' \int \frac{dz}{\iota'} \right) \quad (6.14)$$

modulo the transformation in (2.21).

Recall $\xi = \kappa y + gH(\varpi)$ and $\eta = \tau y$. By (4.1), we have the following theorem.

Theorem 6.1. *Let ι be any function of z and let α be an arbitrary function of t . Suppose that \Im, φ are any functions of $\hat{\varpi} = \alpha - \iota$ and $a \in \mathbb{R}$. We have the following solutions of the three-dimensional classical non-steady boundary layer equations (1.3), (1.4) and (1.5):*

$$w = \frac{\alpha'}{\iota'}, \quad v = \frac{\alpha'\iota''y}{(\iota')^2} - \frac{\alpha''}{2\alpha'}y, \quad (6.15)$$

p is given in (6.14), and

$$\begin{aligned} u &= \frac{\alpha''}{2\alpha'}x + \frac{\varphi}{2\sqrt{\alpha'}} \left[\exp \left(2\nu\Im^2 \int \frac{dz}{(\iota')^2} \right) \right] \\ &\times \left[\exp \left(\frac{\sqrt{\alpha'}\Im y}{\iota'} \right) + a \exp \left(-\frac{\sqrt{\alpha'}\Im y}{\iota'} \right) \right] \end{aligned} \quad (6.16)$$

or

$$u = \frac{\alpha''}{2\alpha'}x - \frac{\varphi}{\sqrt{\alpha'}} \left[\exp \left(-2\nu\Im^2 \int \frac{dz}{(\iota')^2} \right) \right] \cos \left(\frac{\sqrt{\alpha'}\Im y}{\iota'} \right). \quad (6.17)$$

Suppose

$$\kappa = \alpha'(t)x, \quad \tau = \beta'(t)z, \quad \sigma = e^{\alpha+\beta} \quad (6.18)$$

for some functions α and β of t . Equation (6.9) naturally holds. By the compatibility $p_{xz} = p_{zx}$ in (6.7), we take

$$(\sigma g)^2 = \varepsilon(t, z) - \varsigma(t, x) \quad (6.19)$$

for some two-variable functions ε and ς . Now (6.13) is implied by the system of the following equations:

$$\varepsilon_t + \beta' z \varepsilon_z + 2(\alpha' - (-1)^r \nu e^{2(\alpha+\beta)}) \varepsilon = 0, \quad \varsigma_t + \alpha' x \varsigma_x + 2(\alpha' - (-1)^r \nu e^{2(\alpha+\beta)}) \varsigma = 0. \quad (6.20)$$

Thus

$$\varepsilon = \Im(z e^{-\beta}) \exp \left(-2\alpha + (-1)^r 2\nu \int e^{2(\alpha+\beta)} dt \right) \quad (6.21)$$

and

$$\varsigma = \iota(x e^{-\alpha}) \exp \left(-2\alpha + (-1)^r 2\nu \int e^{2(\alpha+\beta)} dt \right), \quad (6.22)$$

where ι and \Im are arbitrary one-variable functions. Hence

$$g = \sqrt{\Im(ze^{-\beta}) - \iota(xe^{-\alpha})} \exp\left(-\beta - 2\alpha + (-1)^r \nu \int e^{2(\alpha+\beta)} dt\right) \quad (6.23)$$

and

$$\begin{aligned} p = & \frac{\rho}{2}[(a\delta_{0,r} + \delta_{1,r})\iota(xe^{-\alpha}) \exp\left((-1)^r 2\nu \int e^{2(\alpha+\beta)} dt - 2\alpha\right) \\ & - (\alpha'' + (\alpha')^2)x^2 - (\beta'' + (\beta')^2)z^2] \end{aligned} \quad (6.24)$$

modulo the transformation in (2.21).

Recall $\xi = \kappa y + gH(\varpi)$ and $\eta = \tau y$. By (4.1), we have the following theorem.

Theorem 6.2. *Let ι, \Im be any one-variable functions and let α, β be any functions of t . We have the following solutions of the three-dimensional classical non-steady boundary layer equations (1.3), (1.4) and (1.5): (1) $w = \beta'z$,*

$$u = \alpha'x + \sqrt{\Im(ze^{-\beta}) - \iota(xe^{-\alpha})} \exp\left(-\alpha - \nu \int e^{2(\alpha+\beta)} dt\right) \cos(e^{\alpha+\beta}y), \quad (6.25)$$

$$\begin{aligned} v = & \frac{e^{-\alpha}\iota'(xe^{-\alpha})}{2\sqrt{\Im(ze^{-\beta}) - \iota(xe^{-\alpha})}} \exp\left(-\beta - 2\alpha - \nu \int e^{2(\alpha+\beta)} dt\right) \\ & \times \sin(e^{\alpha+\beta}y) - (\alpha' + \beta')y \end{aligned} \quad (6.26)$$

and p is given in (6.24) with $r = 1$; (2) $w = \beta'z$,

$$\begin{aligned} u = & \alpha'x + \frac{1}{2}\sqrt{\Im(ze^{-\beta}) - \iota(xe^{-\alpha})} \left[\exp\left(-\alpha + \nu \int e^{2(\alpha+\beta)} dt\right) \right] \\ & \times [\exp(e^{\alpha+\beta}y) + a \exp(-e^{\alpha+\beta}y)] \end{aligned} \quad (6.27)$$

$$\begin{aligned} v = & \frac{e^{-\alpha}\iota'(xe^{-\alpha})}{4\sqrt{\Im(ze^{-\beta}) - \iota(xe^{-\alpha})}} \left[\exp\left(-\beta - 2\alpha + \nu \int e^{2(\alpha+\beta)} dt\right) \right] \\ & \times [\exp(e^{\alpha+\beta}y) - a \exp(-e^{\alpha+\beta}y)] - (\alpha' + \beta')y \end{aligned} \quad (6.28)$$

for $a \in \mathbb{R}$, and p is given in (6.24) with $r = 0$.

Case 6. $H = \Phi = \vartheta_r$ in (6.1) and (6.2), and $\sigma = e^{\alpha(t)}$ for some function α of t .

Note $\varpi = e^{\alpha}y$ in this case. Moreover,

$$\begin{aligned} \xi_y \xi_{yx} - (\xi_x + \eta_z) \xi_{yy} + \eta_y \xi_{yz} = & \kappa \kappa_x + \tau \kappa_z - (-1)^r e^{2\alpha} g[(\kappa_x + \tau_z)y + f_x] \vartheta_r \\ & + e^{\alpha}[(g\kappa)_x + g_z \tau + \zeta \kappa_z] \hat{\vartheta}_r + (a\delta_{0,r} + \delta_{1,r}) e^{2\alpha} g(g_x + \zeta_z) + e^{2\alpha} (\zeta g_z - g \zeta_z) \hat{\vartheta}_r^2 \end{aligned} \quad (6.29)$$

by (4.11) and

$$\xi_{yt} = \kappa_t + (\alpha'g + g_t)e^\alpha \hat{\vartheta}_r + (-1)^r \alpha' e^{2\alpha} g y \vartheta_r, \quad \xi_{yyy} = (-1)^r e^{3\alpha} g \hat{\vartheta}_r. \quad (6.30)$$

By symmetry, (4.2) and (4.3) are implied by the following system of partial differential equations: $f_x = 0$,

$$\alpha' - \kappa_x - \tau_z = 0, \quad (6.31)$$

$$\zeta g_z - g \zeta_z = 0, \quad g \zeta_x - g_x \zeta = 0. \quad (6.32)$$

$$\kappa_t + \kappa \kappa_x + \tau \kappa_z + (a\delta_{0,r} + \delta_{1,r})e^{2\alpha} g(g_x + \zeta_z) + \frac{1}{\rho} p_x = 0, \quad (6.33)$$

$$\tau_t + \kappa \tau_x + \tau \tau_z + (a\delta_{0,r} + \delta_{1,r})e^{2\alpha} \zeta(g_x + \zeta_z) + \frac{1}{\rho} p_z = 0, \quad (6.34)$$

$$\alpha'g + g_t + (g\kappa)_x + g_z \tau + \zeta \kappa_z - (-1)^r \nu e^{2\alpha} g = 0, \quad (6.35)$$

$$\alpha' \zeta + \zeta_t + (\zeta \tau)_z + \zeta_x \kappa + g \tau_x - (-1)^r \nu e^{2\alpha} \zeta = 0. \quad (6.36)$$

We take $f = 0$ by (4.1). Moreover, (6.31) yields

$$\kappa = \alpha'x - \varsigma_z, \quad \tau = \varsigma_x \quad (6.37)$$

for some function ς of t, x, z . By (6.32),

$$\zeta = g \tan \gamma \quad (6.38)$$

for some function γ of t . According to (4.28) and (4.29), (6.35) and (6.36) give

$$\gamma' \sec^2 \gamma + \tau_x - \kappa_z \tan^2 \gamma + (\tau_z - \kappa_x) \tan \gamma = 0, \quad (6.39)$$

equivalently,

$$\gamma' \sec^2 \gamma + \varsigma_{xx} + \varsigma_{zz} \tan^2 \gamma + 2\varsigma_{xz} \tan \gamma - \alpha' \tan \gamma = 0. \quad (6.40)$$

Now we use the notations in (4.31). The above equation is equivalent to

$$\partial_{\tilde{\omega}}^2(\varsigma) = \alpha' \sin \gamma \cos \gamma - \gamma', \quad (6.41)$$

that is,

$$\varsigma = \phi(t, \hat{\omega}) + \psi(t, \hat{\omega}) \tilde{\omega} + \frac{1}{2}(\alpha' \sin \gamma \cos \gamma - \gamma') \tilde{\omega}^2 \quad (6.42)$$

for some functions ϕ and ψ of t and $\hat{\omega}$. So

$$\kappa = \alpha'x - (\phi_{\hat{\omega}} + \tilde{\omega} \psi_{\hat{\omega}}) \cos \gamma - \psi \sin \gamma - (\alpha' \sin \gamma \cos \gamma - \gamma') \tilde{\omega} \sin \gamma, \quad (6.43)$$

$$\tau = -(\phi_{\hat{\omega}} + \tilde{\omega} \psi_{\hat{\omega}}) \sin \gamma + \psi \cos \gamma + (\alpha' \sin \gamma \cos \gamma - \gamma') \tilde{\omega} \cos \gamma. \quad (6.44)$$

Note

$$\tau_x - \kappa_z = \phi_{\hat{\omega}\hat{\omega}} + \tilde{\omega}\psi_{\hat{\omega}\hat{\omega}} + \alpha' \sin \gamma \cos \gamma - \gamma'. \quad (6.45)$$

For simplicity of solving the problem, we assume

$$\phi = -\frac{1}{2}(\alpha' \sin \gamma \cos \gamma - \gamma')\hat{\omega}^2, \quad \psi_{\hat{\omega}\hat{\omega}} = 0, \quad (6.46)$$

that is, $\tau_x - \kappa_z = 0$. The compatibility $p_{xz} = p_{zx}$ in (6.33) and (6.36) is equivalent to $\partial_{\hat{\omega}}\partial_{\tilde{\omega}}(g^2) = 0$. So

$$g^2 = \varepsilon(t, \hat{\omega}) + \varsigma(t, \tilde{\omega}) \quad (6.47)$$

for some function ε of t and $\hat{\omega}$, and some function ς of t and $\tilde{\omega}$. On the other hand, (6.35) can be written as

$$g_t + \kappa g_x + \tau g_z + (\alpha' + \kappa_x + \kappa_z \tan \gamma - (-1)^r \nu e^{2\alpha})g = 0, \quad (6.48)$$

equivalently,

$$\begin{aligned} g_t + \alpha' x \partial_x(g) - (\phi_{\hat{\omega}} + \tilde{\omega}\psi_{\hat{\omega}})\partial_{\tilde{\omega}}(g) + (\psi + (\alpha' \sin \gamma \cos \gamma - \gamma')\tilde{\omega})\partial_{\hat{\omega}}(g) \\ + (\alpha'(2 - \sin^2 \gamma) - \psi_{\hat{\omega}} + \gamma' \tan \gamma - (-1)^r \nu e^{2\alpha})g = 0. \end{aligned} \quad (6.49)$$

By (4.31) and (4.32), we obtain

$$\begin{aligned} g_t + [\tilde{\omega}(\alpha' \cos^2 \gamma - \psi_{\hat{\omega}}) + \gamma' \hat{\omega}]\partial_{\tilde{\omega}}(g) + (\alpha' \hat{\omega} \sin^2 \gamma - \gamma' \tilde{\omega} + \psi)\partial_{\hat{\omega}}(g) \\ + (\alpha'(2 - \sin^2 \gamma) - \psi_{\hat{\omega}} + \gamma' \tan \gamma - (-1)^r \nu e^{2\alpha})g = 0. \end{aligned} \quad (6.50)$$

Multiplying $2g$ to the above equation, we have

$$\begin{aligned} (g^2)_t + [\tilde{\omega}(\alpha' \cos^2 \gamma - \psi_{\hat{\omega}}) + \gamma' \hat{\omega}](g^2)_{\tilde{\omega}} + (\alpha' \hat{\omega} \sin^2 \gamma - \gamma' \tilde{\omega} + \psi)(g^2)_{\hat{\omega}} \\ + 2(\alpha'(2 - \sin^2 \gamma) - \psi_{\hat{\omega}} + \gamma' \tan \gamma - (-1)^r \nu e^{2\alpha})g^2 = 0. \end{aligned} \quad (6.51)$$

Substituting (6.47) into the above equation, we get

$$\begin{aligned} \varepsilon_t + \varsigma_t + [\tilde{\omega}(\alpha' \cos^2 \gamma - \psi_{\hat{\omega}}) + 2\gamma' \hat{\omega}]\varsigma_{\tilde{\omega}} + (\alpha' \hat{\omega} \sin^2 \gamma - 2\gamma' \tilde{\omega} + \psi)\varepsilon_{\hat{\omega}} \\ + 2(\alpha'(2 - \sin^2 \gamma) - \psi_{\hat{\omega}} + \gamma' \tan \gamma - (-1)^r \nu e^{2\alpha})(\varepsilon + \varsigma) = 0. \end{aligned} \quad (6.52)$$

In order to solve (6.52), we assume

$$\varepsilon_{\hat{\omega}} = 2\beta\hat{\omega} + \beta_1, \quad \varsigma_{\tilde{\omega}} = 2\beta\tilde{\omega} + \beta_2 \quad (6.53)$$

for some functions β , β_1 , β_2 of t . So ε is a quadratic polynomial in $\hat{\omega}$ and ς is a quadratic polynomial in $\tilde{\omega}$, whose coefficients are functions of t . Now (6.52) is implied by the following system of partial differential equations:

$$\begin{aligned} & \varepsilon_t + 2\beta_2\gamma'\hat{\omega} + (\alpha'\hat{\omega}\sin^2\gamma + \psi)(2\beta\hat{\omega} + \beta_1) \\ & + 2(\alpha'(2 - \sin^2\gamma) - \psi_{\hat{\omega}} + \gamma'\tan\gamma - (-1)^r\nu e^{2\alpha})\varepsilon = 0, \end{aligned} \quad (6.54)$$

$$\begin{aligned} & \varsigma_t - 2\beta_1\gamma'\tilde{\omega} + [\tilde{\omega}(\alpha'\cos^2\gamma - \psi_{\tilde{\omega}})](2\beta\tilde{\omega} + \beta_2) \\ & + 2(\alpha'(2 - \sin^2\gamma) - \psi_{\tilde{\omega}} + \gamma'\tan\gamma - (-1)^r\nu e^{2\alpha})\varsigma = 0. \end{aligned} \quad (6.55)$$

By the coefficients of quadratic terms, we have

$$\beta' + 2(2\alpha' + \gamma'\tan\gamma - (-1)^r\nu e^{2\alpha})\beta = 0, \quad (6.56)$$

$$\beta' + 2(\alpha'(2 + \cos 2\gamma) - 2\psi_{\hat{\omega}} + \gamma'\tan\gamma - (-1)^r\nu e^{2\alpha})\beta = 0, \quad (6.57)$$

which implies

$$\psi_{\hat{\omega}} = \frac{\alpha'}{2} \cos 2\gamma. \quad (6.58)$$

For simplicity, we take

$$\psi = \frac{\alpha'}{2} \hat{\omega} \cos 2\gamma. \quad (6.59)$$

According (6.56),

$$\beta = b_1 \left[\exp \left((-1)^r 2\nu \int e^{2\alpha} dt - 4\alpha \right) \right] \cos^2 \gamma \quad (6.60)$$

for $b_1 \in \mathbb{R}$.

The coefficients of $\hat{\omega}$ in (6.54) and of $\tilde{\omega}$ in (6.55) give

$$\beta'_1 + 2\beta_2\gamma + (7\alpha'/2 + 2\gamma'\tan\gamma - (-1)^r 2\nu e^{2\alpha})\beta_1 = 0, \quad (6.61)$$

$$\beta'_2 - 2\beta_1\gamma' + (7\alpha'/2 + 2\gamma'\tan\gamma + 2\nu e^{2\alpha})\beta_2 = 0. \quad (6.62)$$

So

$$\beta_1 = (b_2 \cos 2\gamma + b_3 \sin 2\gamma) \left[\exp \left((-1)^r 2\nu \int e^{2\alpha} dt - 7\alpha/2 \right) \right] \cos^2 \gamma, \quad (6.63)$$

$$\beta_2 = (-b_2 \sin 2\gamma + b_3 \cos 2\gamma) \left[\exp \left((-1)^r 2\nu \int e^{2\alpha} dt - 7\alpha/2 \right) \right] \cos^2 \gamma \quad (6.64)$$

for $b_2, b_3 \in \mathbb{R}$. According to (6.47), we take the zero constant term of ς and the constant term of ε to be

$$b_4 \left[\exp \left((-1)^r 2\nu \int e^{2\alpha} dt - 3\alpha \right) \right] \cos^2 \gamma \quad (6.65)$$

by (6.54) with $b_4 \in \mathbb{R}$. Hence we take

$$\begin{aligned} g &= \sqrt{b_1 e^{-\alpha}(\hat{\omega}^2 + \tilde{\omega}^2) + e^{-\alpha/2}[(b_2\hat{\omega} + b_3\tilde{\omega})\cos 2\gamma + (b_3\hat{\omega} - b_2\tilde{\omega})\sin 2\gamma]} + b_4 \\ &\times \left[\exp \left((-1)^r \nu \int e^{2\alpha} dt - 3\alpha/2 \right) \right] \cos \gamma. \end{aligned} \quad (6.66)$$

Furthermore,

$$\kappa = \frac{\alpha' x}{2} - \gamma'(z \cos 2\gamma - x \sin 2\gamma), \quad (6.67)$$

$$\tau = \frac{\alpha' z}{2} - \gamma'(x \cos 2\gamma + z \sin 2\gamma) \quad (6.68)$$

by (4.31), (6.43), (6.44), (6.46) and (6.58).

Set

$$\hat{\kappa} = \frac{\alpha'}{4}(x^2 + z^2) + \frac{\gamma' \sin 2\gamma}{2}(x^2 - z^2) - xz\gamma' \cos 2\gamma. \quad (6.69)$$

Then

$$\kappa = \hat{\kappa}_x, \quad \tau = \hat{\kappa}_z \implies \kappa_t = (\hat{\kappa}_t)_x, \quad \tau_t = (\hat{\kappa}_t)_z. \quad (6.70)$$

Moreover,

$$\begin{aligned} e^{2\alpha} g(g_x + \zeta_z) &= \frac{\sec \gamma}{2} \partial_{\tilde{\omega}}(e^{2\alpha} g^2) \\ &= \frac{\cos \gamma}{2} [2b_1 e^{-\alpha} \tilde{\omega} + e^{-\alpha/2} (b_3 \cos 2\gamma - b_2 \sin 2\gamma)] \left[\exp \left((-1)^r 2\nu \int e^{2\alpha} dt - \alpha \right) \right]. \end{aligned} \quad (6.71)$$

By (6.33), (6.34) and (6.69)-(6.71),

$$\begin{aligned} p &= \rho \{ xz(\gamma'' \cos 2\gamma + \alpha' \gamma' \cos 2\gamma - 2(\gamma')^2 \sin 2\gamma) + \frac{\gamma'' \sin 2\gamma + 2(\gamma')^2 \cos \gamma}{2} (z^2 - x^2) \\ &\quad - \frac{4(\gamma')^2 + 2\alpha'' + (\alpha')^2}{8} (x^2 + z^2) - \frac{a\delta_{0,r} + \delta_{1,r}}{2} \left[\exp \left((-1)^r 2\nu \int e^{2\alpha} dt - \alpha \right) \right] \\ &\quad \times [b_1 e^{-\alpha} \tilde{\omega}^2 + e^{-\alpha/2} (b_3 \cos 2\gamma - b_2 \sin 2\gamma) \tilde{\omega}] \} \end{aligned} \quad (6.72)$$

modulo the transformation in (2.21).

Recall $\xi = \kappa y + gH(\varpi)$ and $\eta = \tau y + \zeta \Phi$. Moreover, $\hat{\omega}^2 + \tilde{\omega}^2 = x^2 + z^2$ according to (4.31). By (4.1), we have the following theorem.

Theorem 6.3. *Let α, γ be any functions of t and let $a, b_1, b_2, b_3, b_4 \in \mathbb{R}$. For $r = 0, 1$, we define ϑ_r and $\hat{\vartheta}_r$ in (6.1) and (6.2) with $\varpi = e^\alpha y$, and $\hat{\omega}$ and $\tilde{\omega}$ in (4.31). We have the following solutions of the three-dimensional classical non-steady boundary layer equations (1.3), (1.4) and (1.5):*

$$\begin{aligned} u &= \frac{\alpha' x}{2} + \sqrt{b_1 e^{-\alpha} (x^2 + z^2) + e^{-\alpha/2} [(b_2 \hat{\omega} + b_3 \tilde{\omega}) \cos 2\gamma + (b_3 \hat{\omega} - b_2 \tilde{\omega}) \sin 2\gamma] + b_4} \\ &\quad \times \left[\exp \left((-1)^r \nu \int e^{2\alpha} dt - \alpha/2 \right) \right] \hat{\vartheta}_r \cos \gamma - \gamma'(z \cos 2\gamma - x \sin 2\gamma), \end{aligned} \quad (6.73)$$

$$\begin{aligned} w &= \frac{\alpha' z}{2} + \sqrt{b_1 e^{-\alpha} (x^2 + z^2) + e^{-\alpha/2} [(b_2 \hat{\omega} + b_3 \tilde{\omega}) \cos 2\gamma + (b_3 \hat{\omega} - b_2 \tilde{\omega}) \sin 2\gamma] + b_4} \\ &\quad \times \left[\exp \left((-1)^r \nu \int e^{2\alpha} dt - \alpha/2 \right) \right] \hat{\vartheta}_r \sin \gamma - \gamma'(x \cos 2\gamma + z \sin 2\gamma), \end{aligned} \quad (6.74)$$

$$\begin{aligned}
v &= -\frac{2b_1e^{-\alpha}\tilde{\omega} + e^{-\alpha/2}(b_3\cos 2\gamma - b_2\sin 2\gamma)}{2\sqrt{b_1e^{-\alpha}(x^2 + z^2) + e^{-\alpha/2}[(b_2\hat{\omega} + b_3\tilde{\omega})\cos 2\gamma + (b_3\hat{\omega} - b_2\tilde{\omega})\sin 2\gamma] + b_4}} \\
&\times \\
&\times \left[\exp\left((-1)^r\nu \int e^{2\alpha}dt - 3\alpha/2\right) \right] \vartheta_r - \alpha'y
\end{aligned} \tag{6.75}$$

and p is given in (6.72).

Case 7. $H = \vartheta_r$ and $\Phi = \hat{\vartheta}_r$ in (6.1) and (6.2) with $r = 0, 1$.

In this case,

$$\begin{aligned}
&\xi_y\xi_{yx} - (\xi_x + \eta_z)\xi_{yy} + \eta_y\xi_{yz} \\
&= \kappa\kappa_x + \tau\kappa_z + (a\delta_{0,r} + \delta_{1,r})\sigma^2gg_x + [(\sigma g\kappa)_x + (\sigma g)_z\tau]\hat{\vartheta}_r + \sigma\sigma_xg^2\hat{\vartheta}_r^2 + (-1)^r\sigma\{\zeta\kappa_z \\
&\quad + g[(\kappa\sigma_x + \tau\sigma_z - \sigma(\kappa_x + \tau_z))y - \sigma f_x]\}\vartheta_r + (-1)^r[\sigma\zeta(\sigma g)_z - \sigma^2g\zeta_z]\vartheta_r\hat{\vartheta}_r
\end{aligned} \tag{6.76}$$

by (4.11), and

$$\begin{aligned}
&\xi_y\eta_{yx} - (\xi_x + \eta_z)\eta_{yy} + \eta_y\eta_{yz} \\
&= \kappa\tau_x + \tau\tau_z + (-1)^r\{[(\sigma\zeta\tau)_z + (\sigma\zeta)_x\kappa]\vartheta_r - (a\delta_{0,r} + \delta_{1,r})\sigma^2\zeta\zeta_z\} + \sigma\sigma_z\zeta^2\vartheta_r^2 + \sigma\{g\tau_x \\
&\quad + (-1)^r\zeta[(\kappa\sigma_x + \tau\sigma_z - \sigma(\kappa_x + \tau_z))y - \sigma f_x] + (-1)^r[g(\sigma\zeta)_x - \sigma g_x\zeta]\}\hat{\vartheta}_r
\end{aligned} \tag{6.77}$$

by (4.12). Moreover,

$$\xi_{yt} = \kappa_t + (\sigma g)_t\hat{\vartheta}_r + (-1)^r\sigma\sigma_tgy\vartheta_r, \quad \xi_{yyy} = (-1)^r\sigma^3g\hat{\vartheta}_r, \tag{6.78}$$

$$\eta_{yt} = \tau_t + (-1)^r(\sigma\zeta)_t\vartheta_r + (-1)^r\sigma\sigma_t\zeta y\hat{\vartheta}_r, \quad \eta_{yyy} = \sigma^3\zeta\vartheta_r \tag{6.79}$$

by (4.4) and (4.5). Thus (4.2) and (4.3) are implied by the following system of partial differential equations:

$$\sigma_x = \sigma_z = 0, \quad \zeta\kappa_z - \sigma gf_x = 0, \quad g\tau_x - (-1)^r\sigma\zeta f_x = 0, \tag{6.80}$$

$$\kappa_t + \kappa\kappa_x + \tau\kappa_z + (a\delta_{0,r} + \delta_{1,r})\sigma^2gg_x + \frac{1}{\rho}p_x = 0, \tag{6.81}$$

$$\tau_t + \kappa\tau_x + \tau\tau_z - (-1)^r(a\delta_{0,r} + \delta_{1,r})\sigma^2\zeta\zeta_z + \frac{1}{\rho}p_z = 0, \tag{6.82}$$

$$(\sigma g)_t + (\sigma g\kappa)_x + (\sigma g)_z\tau - (-1)^r\nu\sigma^3g = 0, \tag{6.83}$$

$$(\sigma\zeta)_t + (\sigma\zeta\tau)_z + (\sigma\zeta)_x\kappa - (-1)^r\nu\sigma^3\zeta = 0, \tag{6.84}$$

$$\sigma_t - \sigma(\kappa_x + \tau_z) = 0, \tag{6.85}$$

$$\zeta g_z - g\zeta_z = 0, \quad g\zeta_x - g_x\zeta = 0. \tag{6.86}$$

By (6.80) and (6.86), we have

$$\sigma = e^\alpha, \quad \zeta = g \tan \gamma, \quad \tau_x \cot \gamma - (-1)^r \kappa_z \tan \gamma = 0. \quad (6.87)$$

Moreover, (6.83) and (6.84) become

$$g_t + \kappa g_x + \tau g_z + (\alpha' + \kappa_x - (-1)^r \nu e^{2\alpha})g = 0, \quad (6.88)$$

$$\gamma' g \sec^2 \gamma + (g_t + \kappa g_x + \tau g_z + (\alpha' + \tau_z - (-1)^r \nu e^{2\alpha})g) \tan \gamma = 0. \quad (6.89)$$

Furthermore, $[(6.89) - \tan \gamma \times (6.88)]/g$ yields

$$\gamma' \sec^2 \gamma + (\tau_z - \kappa_x) \tan \gamma = 0 \implies \kappa_x - \tau_z = 2\gamma' \csc 2\gamma. \quad (6.90)$$

On the other hand, (6.85) says

$$\kappa_x + \tau_z = \alpha'. \quad (6.91)$$

Thus

$$\kappa_x = \frac{\alpha'}{2} + \gamma' \csc 2\gamma, \quad \tau_z = \frac{\alpha'}{2} - \gamma' \csc 2\gamma. \quad (6.92)$$

We take

$$\kappa = \left(\frac{\alpha'}{2} + \gamma' \csc 2\gamma \right) x, \quad \tau = \left(\frac{\alpha'}{2} - \gamma' \csc 2\gamma \right) z \quad (6.93)$$

for simplicity. So the third equation in (6.87) naturally holds. Moreover, the compatibility $p_{xz} = p_{zx}$ in (6.81) and (6.82) is implied by $(g^2)_{xz} = 0$. So

$$g^2 = \phi(t, x) + \psi(t, z) \quad (6.94)$$

for some two-variable functions ϕ and ψ . By (6.93), (6.88) becomes

$$\begin{aligned} (g^2)_t + \left(\frac{\alpha'}{2} + \gamma' \csc 2\gamma \right) x (g^2)_x + \left(\frac{\alpha'}{2} - \gamma' \csc 2\gamma \right) z (g^2)_z \\ + (3\alpha' + 2\gamma' \csc 2\gamma - (-1)^r 2\nu e^{2\alpha})g^2 = 0. \end{aligned} \quad (6.95)$$

Thus

$$g^2 = e^{-3\alpha + (-1)^r 2\nu \int e^{2\alpha} dt} (\iota(xe^{-\alpha/2} \sqrt{\cot \gamma}) + \Im(z e^{-\alpha/2} \sqrt{\tan \gamma})) \cot \gamma \quad (6.96)$$

for some one-variable functions ι, \Im . According (6.81), (6.82), (6.87), (6.93) and (6.96), we have:

$$\begin{aligned} p &= \frac{\rho}{2} \{ (a\delta_{0,r} + \delta_{1,r}) [(-1)^r (\tan \gamma) \Im(z e^{-\alpha/2} \sqrt{\tan \gamma}) - (\cot \gamma) \iota(xe^{-\alpha/2} \sqrt{\cot \gamma})] \\ &\quad \times e^{-\alpha + (-1)^r 2\nu \int e^{2\alpha} dt} - \left[\left(\frac{\alpha'}{2} + \gamma' \csc 2\gamma \right)^2 + \frac{\alpha''}{2} + \csc 2\gamma (\gamma'' - 2(\gamma')^2 \cot 2\gamma) \right] x^2 \\ &\quad - \left[\left(\frac{\alpha'}{2} - \gamma' \csc 2\gamma \right)^2 + \frac{\alpha''}{2} - \csc 2\gamma (\gamma'' - 2(\gamma')^2 \cot 2\gamma) \right] z^2 \} \end{aligned} \quad (6.97)$$

modulo the transformation in (2.21). Furthermore, we take $f = 0$ by (6.80).

Note $\xi = \kappa y + g\vartheta_r$ and $\eta = \tau y + \zeta\hat{\vartheta}_r$. By (4.1), we have the following theorem.

Theorem 6.4. *Let ι, \Im be any one-variable functions and let α, γ be any functions of t . For $r = 0, 1$, we define ϑ_r and $\hat{\vartheta}_r$ in (6.1) and (6.2) with $\varpi = e^\alpha y$ and $a \in \mathbb{R}$. We have the following solutions of the three-dimensional classical non-steady boundary layer equations (1.3), (1.4) and (1.5):*

$$u = \left(\frac{\alpha'}{2} + \gamma' \csc 2\gamma \right) x + e^{-\alpha/2 + (-1)^r \nu \int e^{2\alpha} dt} \hat{\vartheta}_r \\ \times \sqrt{(\iota(xe^{-\alpha/2} \sqrt{\cot \gamma}) + \Im(ze^{-\alpha/2} \sqrt{\tan \gamma})) \cot \gamma}, \quad (6.98)$$

$$w = \left(\frac{\alpha'}{2} - \gamma' \csc 2\gamma \right) z + (-1)^r e^{-\alpha/2 + (-1)^r \nu \int e^{2\alpha} dt} \vartheta_r \\ \times \sqrt{(\iota(xe^{-\alpha/2} \sqrt{\cot \gamma}) + \Im(ze^{-\alpha/2} \sqrt{\tan \gamma})) \tan \gamma}, \quad (6.99)$$

$$v = -\alpha' y - \frac{e^{-2\alpha + (-1)^r \nu \int e^{2\alpha} dt}}{2\sqrt{(\iota(xe^{-\alpha/2} \sqrt{\cot \gamma}) + \Im(ze^{-\alpha/2} \sqrt{\tan \gamma}))}} \\ \times [(\cot \gamma) \vartheta_r \iota'(xe^{-\alpha/2} \sqrt{\cot \gamma}) + (\tan \gamma) \hat{\vartheta}_r \Im'(ze^{-\alpha/2} \sqrt{\tan \gamma})] \quad (6.100)$$

and p is given in (6.97).

7 3-D Rational Approach

In this section, we will find certain function-parameter exact solutions of rational in y for the three-dimensional classical non-steady boundary layer equations (1.3), (1.4) and (1.5).

First we consider another three cases of solving the three-dimensional classical non-steady boundary layer equations (1.3), (1.4) and (1.5) based on the settings in (4.1)-(4.12).

Case 8. $H = \Phi = y^{-1}$.

In this case $\sigma = 1$ and $\varpi = y$. So (4.11) becomes

$$\xi_y \xi_{yx} - (\xi_x + \eta_z) \xi_{yy} + \eta_y \xi_{yz} = \kappa \kappa_x + \tau \kappa_z - [\zeta \kappa_z + (g\kappa)_x + g_z \tau] y^{-2} \\ - 2g[(\kappa_x + \tau_z)y + f_x] y^{-3} + (\zeta g_z - g g_x - 2g \zeta_z) y^{-4}. \quad (7.1)$$

Moreover,

$$\xi_{yt} = \tau_t - g_t y^{-2}, \quad \xi_{yyy} = -6g y^{-4}. \quad (7.2)$$

By symmetry, (4.2) and (4.3) are implied by the following system of partial differential equations: $f_x = 0$,

$$\kappa_t + \kappa\kappa_x + \tau\kappa_z + \frac{1}{\rho}p_x = 0, \quad \tau_t + \kappa\tau_x + \tau\tau_z + \frac{1}{\rho}p_z = 0, \quad (7.3)$$

$$\zeta g_z - gg_x - 2g\zeta_z + 6\nu g = 0, \quad g\zeta_x - \zeta\zeta_z - 2\zeta g_x + 6\nu\zeta = 0, \quad (7.4)$$

$$g_t + \zeta\kappa_z + \kappa g_x + g_z\tau + g(3\kappa_x + 2\tau_z) = 0, \quad (7.5)$$

$$\zeta_t + g\tau_x + \kappa\zeta_x + \tau\zeta_z + \zeta(3\tau_z + 2\kappa_x) = 0. \quad (7.6)$$

We take $f = 0$ and assume $\zeta = g \tan \gamma$ for some function γ of t . Use the notations in (4.31). Then both equations in (7.4) are equivalent to the equation:

$$g_x + g_z \tan \gamma = 6\nu \implies g = 6\nu x + \phi(t, \hat{\omega}) \quad (7.7)$$

for some two variable function ϕ . Moreover, (7.6) $-\tan \gamma \times$ (7.5) is implied by

$$\gamma' \sec^2 \gamma + \tau_x - \kappa_z \tan^2 \gamma + (\tau_z - \kappa_x) \tan \gamma = 0, \quad (7.8)$$

equivalently,

$$\gamma' \sec \gamma + \partial_{\hat{\omega}}(\tau - \kappa \tan \gamma) = 0 \quad (7.9)$$

by (4.32). So

$$\tau = \kappa \tan \gamma + \psi(t, \hat{\omega}) - \gamma' \hat{\omega} \sec \gamma \quad (7.10)$$

for another two-variable function ψ . To simplify the problem, we assume

$$\kappa_z - \tau_x = 0 \sim \partial_{\hat{\omega}}(\kappa) \sec \gamma = -\psi_{\hat{\omega}} \sin \gamma - \gamma' \quad (7.11)$$

by (4.31) and (4.32). We take

$$\kappa = \alpha \hat{\omega} - \frac{1}{2} \psi \sin 2\gamma - \gamma' \hat{\omega} \cos \gamma \quad (7.12)$$

for some function α of t .

Note that (7.5) becomes

$$\begin{aligned} & \phi_t - \gamma' \hat{\omega} \phi_{\hat{\omega}} + 6\nu \alpha \hat{\omega} - 3\nu \psi \sin 2\gamma - 6\nu \gamma' \hat{\omega} \cos \gamma + (\psi - \gamma' \hat{\omega} \sec \gamma) \phi_{\hat{\omega}} \cos \gamma \\ & + (6\nu x + \phi)(3\alpha \sec \gamma + 2\psi_{\hat{\omega}} \cos \gamma - 2\gamma' \tan \gamma) = 0, \end{aligned} \quad (7.13)$$

equivalently,

$$\begin{aligned} & \phi_t - 3\nu \psi \sin 2\gamma + (\phi - 6\nu \hat{\omega} \sin \gamma)(3\alpha \sec \gamma + 2\psi_{\hat{\omega}} \cos \gamma - 2\gamma' \tan \gamma) - 6\nu \gamma' \hat{\omega} \cos \gamma \\ & + \phi_{\hat{\omega}} \psi \cos \gamma + 2\hat{\omega}(12\nu \alpha - \gamma' \phi_{\hat{\omega}} + 6\nu \psi_{\hat{\omega}} \cos^2 \gamma - 6\nu \gamma' \sin \gamma) = 0. \end{aligned} \quad (7.14)$$

First we have:

$$12\nu\alpha - \gamma'\phi_{\hat{\omega}} + 6\nu\psi_{\hat{\omega}}\cos^2\gamma - 6\nu\gamma'\sin\gamma = 0. \quad (7.15)$$

So we take

$$\psi = \frac{1}{6\nu}\gamma'\phi\sec^2\gamma + \gamma'\hat{\omega}\tan\gamma\sec\gamma - 2\alpha\hat{\omega}\sec^2\gamma. \quad (7.16)$$

Now (7.14) becomes

$$\begin{aligned} \phi_t + (\cos\gamma\phi_{\hat{\omega}} - 3\nu\sin 2\gamma) \left(\frac{1}{6\nu}\gamma'\phi\sec^2\gamma + \gamma'\hat{\omega}\tan\gamma\sec\gamma - 2\alpha\hat{\omega}\sec^2\gamma \right) \\ - 6\nu\gamma'\hat{\omega}\cos\gamma + \frac{1}{3\nu}\gamma'\phi_{\hat{\omega}}(\phi - 6\nu\hat{\omega}\sin\gamma)\sec\gamma = 0, \end{aligned} \quad (7.17)$$

equivalently.

$$\begin{aligned} \phi_t + \frac{1}{2\nu}\gamma'\phi\phi_{\hat{\omega}}\sec\gamma - \gamma'\phi\tan\gamma - (\gamma'\tan\gamma + 2\alpha\sec\gamma)\hat{\omega}\phi_{\hat{\omega}} \\ + 6\nu(2\alpha\tan\gamma - \gamma'\sec\gamma)\hat{\omega} = 0. \end{aligned} \quad (7.18)$$

For simplicity, we take

$$\phi = \beta\hat{\omega} \quad (7.19)$$

for a function $\beta \neq 6\sin\gamma$ of t . Then (7.18) gives

$$\alpha = \frac{\beta^2\gamma' + 2\nu(\beta'\cos\gamma - 2\beta\gamma'\sin\gamma - 6\gamma')}{4\nu(\beta - 6\nu\sin\gamma)}. \quad (7.20)$$

In summary, we have

$$g = 6\nu x + \beta\hat{\omega}, \quad \zeta = (6\nu x + \beta\hat{\omega})\tan\gamma, \quad (7.21)$$

$$\begin{aligned} \kappa &= \alpha\tilde{\omega} - \left(\frac{1}{6\nu}\gamma'\beta\tan\gamma + \gamma'\sin\gamma\tan\gamma - 2\alpha\tan\gamma \right) \hat{\omega} - \gamma'\hat{\omega}\cos\gamma \\ &= \frac{\beta^2\gamma' + 2\nu(\beta'\cos\gamma - 2\beta\gamma'\sin\gamma - 6\gamma')}{4\nu(\beta - 6\nu\sin\gamma)}(\tilde{\omega} + 2\hat{\omega}\tan\gamma) \\ &\quad - \left(\frac{1}{6\nu}\beta\gamma'\tan\gamma + \gamma'\sec\gamma \right) \hat{\omega}, \end{aligned} \quad (7.22)$$

$$\begin{aligned} \tau &= \frac{\beta^2\gamma' + 2\nu(\beta'\cos\gamma - 2\beta\gamma'\sin\gamma - 6\gamma')}{4\nu(\beta - 6\nu\sin\gamma)}(\tilde{\omega}\tan\gamma - 2\hat{\omega}) \\ &\quad + \frac{1}{6\nu}\beta\gamma'\hat{\omega} - \gamma'\tilde{\omega}\sec\gamma. \end{aligned} \quad (7.23)$$

Set

$$\begin{aligned} \hat{\kappa} &= \frac{\beta^2\gamma' + 2\nu(\beta'\cos\gamma - 2\beta\gamma'\sin\gamma - 6\gamma')}{8\nu\cos\gamma(\beta - 6\nu\sin\gamma)}(\tilde{\omega}^2 - 2\hat{\omega}^2) \\ &\quad + \frac{1}{12\nu}\beta\gamma'\sec\gamma\hat{\omega}^2 + \frac{\gamma'}{2}\tan\gamma(\hat{\omega}^2 - \tilde{\omega}^2) - \gamma'\hat{\omega}\tilde{\omega}. \end{aligned} \quad (7.24)$$

Then

$$\kappa = \hat{\kappa}_x, \quad \tau = \hat{\kappa}_z. \quad (7.25)$$

By the fact $\kappa_z = \tau_x$, (7.11) and (7.24), we have

$$p = -\rho\hat{\kappa}_t - \frac{\rho}{2}(\kappa^2 + \tau^2) \quad (7.26)$$

modulo the transformation in (2.21).

In this case, $\xi = \kappa y + gy^{-1}$ and $\eta = \tau y + \zeta y^{-1}$. By (4.1), we have the following theorem.

Theorem 7.1. *Let β, γ be any functions of t . In terms of the notations in (4.31), we have the following solutions of the three-dimensional classical non-steady boundary layer equations (1.3), (1.4) and (1.5):*

$$\begin{aligned} u = & \frac{\beta^2\gamma' + 2\nu(\beta'\cos\gamma - 2\beta\gamma'\sin\gamma - 6\gamma')}{4\nu(\beta - 6\nu\sin\gamma)}(\tilde{\omega} + 2\hat{\omega}\tan\gamma) \\ & - \left(\frac{1}{6\nu}\beta\gamma'\tan\gamma + \gamma'\sec\gamma \right) \hat{\omega} - (6\nu x + \beta\hat{\omega})y^{-2}, \end{aligned} \quad (7.27)$$

$$\begin{aligned} w = & \frac{\beta^2\gamma' + 2\nu(\beta'\cos\gamma - 2\beta\gamma'\sin\gamma - 6\gamma')}{4\nu(\beta - 6\nu\sin\gamma)}(\tilde{\omega}\tan\gamma - 2\hat{\omega}) \\ & + \frac{1}{6\nu}\beta\gamma'\hat{\omega} - \gamma'\tilde{\omega}\sec\gamma - (6\nu x + \beta\hat{\omega})y^{-2}\tan\gamma, \end{aligned} \quad (7.28)$$

$$v = \frac{\beta^2\gamma' + 2\nu(\beta'\cos\gamma - 2\beta\gamma'\sin\gamma - 6\gamma')}{4\nu(\beta - 6\nu\sin\gamma)\cos\gamma}y - \frac{\beta\gamma'}{6\nu}y\sec\gamma - 6\nu y^{-1} \quad (7.29)$$

and p is given in (7.26) via (7.22)-(7.24).

Case 9. $H = \Phi = 0$.

In this case, f can be any function of t, x, z , and (4.2) and (4.3) are equivalent to the equations in (7.3), whose compatibility $p_{xz} = p_{zx}$ is equivalent to (4.30). The simplest solutions are

$$\kappa = \theta_x(t, x, z), \quad \tau = \theta_z(t, x, z) \quad (7.30)$$

for some three-variable function θ . Since $\kappa_z = \tau_x$,

$$p = -\rho\theta_t - \frac{\rho}{2}(\theta_x^2 + \theta_z^2) \quad (7.31)$$

modulo the transformation in (2.21). Besides, the κ and τ in Cases 1,3,4 and 8 work for this case.

Next we assume

$$\kappa = e^{2\alpha}z + h_x(t, x, z), \quad \tau = h_z(t, x, z), \quad (7.32)$$

where α is a nonzero function of t and h is a function of t, x, z . Now (4.30) becomes

$$2\alpha' + h_{xx} + h_{zz} = 0. \quad (7.33)$$

Hence

$$h = \Upsilon_x - \alpha'x^2 \quad (7.34)$$

for some time-dependant harmonic function $\Upsilon(t, x, z)$, that is,

$$\Upsilon_{xx} + \Upsilon_{zz} = 0. \quad (7.35)$$

In this subcase,

$$\kappa = \Upsilon_{xx} - 2\alpha'x + e^{2\alpha}z, \quad \tau = \Upsilon_{xz}. \quad (7.36)$$

$$\kappa_t = \Upsilon_{xxt} - 2\alpha''x + 2\alpha'e^{2\alpha}z, \quad (7.37)$$

$$\kappa_x = \Upsilon_{xxx} - 2\alpha', \quad \kappa_z = \Upsilon_{xxz} + e^{2\alpha}. \quad (7.38)$$

Thus

$$\begin{aligned} \kappa_t + \kappa\kappa_x + \tau\kappa_z &= \Upsilon_{xxt} - 2\alpha''x + 2\alpha'e^{2\alpha}z \\ &+ (\Upsilon_{xx} - 2\alpha'x + e^{2\alpha}z)(\Upsilon_{xxx} - 2\alpha') + \Upsilon_{xz}(\Upsilon_{xxz} + e^{2\alpha}) \\ &= \Upsilon_{xtx} + \frac{1}{2}(\Upsilon_{xx}^2 + \Upsilon_{xz}^2)_x - 2\alpha'(x\Upsilon_{xx})_x + 4(\alpha')^2x + e^{2\alpha}(z\Upsilon_{xxx} + \Upsilon_{zx}) - 2\alpha''x \end{aligned} \quad (7.39)$$

and

$$\tau_t + \kappa\tau_x + \tau\tau_z = \Upsilon_{xzt} + (\Upsilon_{xx} - 2\alpha'x + e^{2\alpha}z)\Upsilon_{xxz} + \Upsilon_{xz}\Upsilon_{xzz}. \quad (7.40)$$

So

$$p = -\rho \left[\Upsilon_{xt} + \frac{1}{2}(\Upsilon_{xx}^2 + \Upsilon_{xz}^2) - 2\alpha'x\Upsilon_{xx} + (2(\alpha')^2 - \alpha'')x^2 + e^{2\alpha}(z\Upsilon_{xx} + \Upsilon_z) \right] \quad (7.41)$$

modulo the transformation in (2.21).

Denote

$$\varpi = x^2 + z^2. \quad (7.42)$$

$$\kappa = \alpha'x + z\phi(t, \varpi), \quad \tau = \alpha'z - x\phi(t, \varpi) \quad (7.43)$$

for some functions α of t and ϕ of t and ϖ . Note

$$\kappa_z - \tau_x = 2(\varpi\phi)_\varpi. \quad (7.44)$$

Then (4.30) becomes

$$(\varpi\phi)_{\varpi t} + 2\alpha'[(\varpi\phi)_{\varpi} + \varpi(\varpi\phi)_{\varpi\varpi}] = 0. \quad (7.45)$$

Thus

$$\phi = \frac{\Im(e^{-2\alpha}\varpi) + \beta}{\varpi} \quad (7.46)$$

for a function β of t and a one-variable function \Im . So

$$\kappa = \alpha'x + \frac{z(\Im(e^{-2\alpha}\varpi) + \beta)}{\varpi}, \quad \tau = \alpha'z - \frac{x(\Im(e^{-2\alpha}\varpi) + \beta)}{\varpi}. \quad (7.47)$$

Moreover,

$$\kappa_t + \kappa\kappa_x + \tau\kappa_z = (\alpha'' + (\alpha')^2)x + \frac{\beta'z}{\varpi} - x\frac{(\Im(e^{-2\alpha}\varpi) + \beta)^2}{\varpi^2}, \quad (7.48)$$

$$\tau_t + \kappa\tau_x + \tau\tau_z = (\alpha'' + (\alpha')^2)z - \frac{\beta'x}{\varpi} - z\frac{(\Im(e^{-2\alpha}\varpi) + \beta)^2}{\varpi^2}. \quad (7.49)$$

Hence

$$p = \frac{\rho}{2} \left\{ e^{-2\alpha} \int \frac{(\Im(e^{-2\alpha}(x^2 + z^2)) + \beta)^2}{[e^{-2\alpha}(x^2 + z^2)]^2} d[e^{-2\alpha}(x^2 + z^2)] - (\alpha'' + (\alpha')^2)(x^2 + z^2) \right\} + \rho\beta' \arctan \frac{z}{x} \quad (7.50)$$

modulo the transformation in (2.21).

The final subcase we are concerned with is to assume

$$\tau = \varepsilon(t, z) \quad (7.51)$$

for some two-variable function ε . In this subcase, (4.30) is equivalent to

$$\partial_z(\kappa_t + \kappa\kappa_x + \tau\kappa_z) = 0. \quad (7.52)$$

We look for a solution of the form

$$\kappa = \phi(t, z) + \psi(t, z)x \quad (7.53)$$

for some two-variable function ϕ and ψ . Note

$$\kappa_t + \kappa\kappa_x + \tau\kappa_z = \phi_t + \psi_t x + \psi\phi + \psi^2 x + \varepsilon_z \phi_z + \varepsilon_z \psi_z x. \quad (7.54)$$

So (7.48) is equivalent to the following system of partial differential equations:

$$\partial_z(\phi_t + \psi\phi + \varepsilon\phi_z) = 0, \quad \partial_z(\psi_t + \psi^2 + \varepsilon\psi_z) = 0. \quad (7.55)$$

To solve the above system, we assume

$$\varepsilon = \frac{\alpha}{\psi_z} - \frac{\varsigma_t(t, z)}{\varsigma_z(t, z)} \quad (7.56)$$

for some functions α of t , and ς of t and z . We have the following solution of the above second equation:

$$\psi = \frac{1}{t + \Im(\varsigma)} \quad (7.57)$$

for another one-variable function \Im . By the first equation,

$$\phi = \frac{b + \delta_{\alpha,0}\Im_1(\varsigma)}{t + \Im(\varsigma)} \quad (7.58)$$

for another one-variable function \Im_1 and a real constant b . In this subcase,

$$\kappa = \frac{b + \delta_{\alpha,0}\Im_1(\varsigma) + x}{t + \Im(\varsigma)}, \quad \tau = -\frac{\alpha(t + \Im(\varsigma))^2}{\varsigma_z \Im'(\varsigma)} - \frac{\varsigma_t}{\varsigma_z}. \quad (7.59)$$

By (7.3) and (7.54),

$$\begin{aligned} p = & -\rho \left[\frac{1}{2} \left(\alpha x^2 + \left(\frac{\alpha(t + \Im(\varsigma))^2}{\varsigma_z \Im'(\varsigma)} + \frac{\varsigma_t}{\varsigma_z} \right)^2 \right) + \int \left[\frac{\varsigma_t \varsigma_{zt} - \varsigma_{tt} \varsigma_z}{\varsigma_z^2} - \frac{\alpha \varsigma_{zt} (t + \Im(\varsigma))^2}{\varsigma_z^2 \Im'(\varsigma)} \right] dz \right. \\ & \left. + \int \frac{(t + \Im(\varsigma))^2 (\alpha \varsigma_t \Im''(\varsigma) - \alpha' \Im'(\varsigma)) - 2\alpha \Im'(\varsigma) (1 + \varsigma_t \Im'(\varsigma))}{\varsigma_z (\Im'(\varsigma))^2} dz + b\alpha x \right] \end{aligned} \quad (7.60)$$

modulo the transformation in (2.21).

Recall $\xi = f + \kappa y$ and $\eta = \tau y$. By (4.1), we have the following theorem.

Theorem 7.2. *Let μ, θ be arbitrary function of t, x, z and let \Im, \Im_1 be any one-variable functions. Suppose that α, β are any functions of t , ς is any function of t and z , b is a real constant and $\Upsilon(t, x, z)$ is a time-dependent harmonic function in x, z . Then we have the following solution of the three-dimensional classical non-steady boundary layer equations (1.3), (1.4) and (1.5): (1)*

$$u = \theta_x, \quad w = \theta_z, \quad v = -(\theta_{xx} + \theta_{zz})y + \mu \quad (7.61)$$

and p is given in (7.31); (2)

$$u = \Upsilon_{xx} - 2\alpha'x + e^{2\alpha}z, \quad w = \Upsilon_{xz}, \quad v = \mu + 2\alpha'y \quad (7.62)$$

and p is given in (7.41); (3)

$$u = \alpha'x + \frac{z(\Im(e^{-2\alpha}(x^2 + z^2)) + \beta)}{x^2 + z^2}, \quad v = \mu - 2\alpha'y, \quad (7.63)$$

$$w = \alpha'z - \frac{x(\Im(e^{-2\alpha}(x^2 + z^2)) + \beta)}{x^2 + z^2}, \quad (7.64)$$

and p is given (7.50); (4)

$$u = \frac{(b + \delta_{\alpha,0})\Im_1(\varsigma) + x}{t + \Im(\varsigma)}, \quad w = -\frac{\alpha(t + \Im(\varsigma))^2}{\varsigma_z \Im'(\varsigma)} - \frac{\varsigma_t}{\varsigma_z}, \quad (7.65)$$

$$\begin{aligned}
v &= \frac{2\alpha\varsigma_z\mathfrak{I}'(\varsigma)(t+\mathfrak{I}(\varsigma))y}{\varsigma_z\mathfrak{I}'(\varsigma)} - \frac{\alpha(t+\mathfrak{I}(\varsigma))^2(\varsigma_{zz}+\varsigma_z^2\mathfrak{I}''(\varsigma))y}{(\varsigma_z\mathfrak{I}'(\varsigma))^2} \\
&+ \mu + \frac{\varsigma_{tz}\varsigma_z - \varsigma_t\varsigma_{zz}}{(\varsigma_z)^2}y - \frac{y}{t+\mathfrak{I}(\varsigma)}
\end{aligned} \tag{7.66}$$

and p is given in (7.60).

Case 10. $H = \Phi = y^2$.

In this case, $\sigma = 1$. Now (4.11) becomes

$$\begin{aligned}
&\xi_y\xi_{yx} - (\xi_x + \eta_z)\xi_{yy} + \eta_y\xi_{yz} = \kappa_t + \kappa\kappa_x + \tau\kappa_z - 2gf_x \\
&+ 2[(g\kappa)_x + g_z\tau + \zeta\kappa_z - g(\kappa_x + \tau_z)]y + 2(gg_x + 2\zeta g_z - g\zeta_z)y^2.
\end{aligned} \tag{7.67}$$

Moreover,

$$\xi_{yt} = \kappa_t + 2g_ty, \quad \xi_{yyy} = 0. \tag{7.68}$$

Modulo the transformation (2.22) and (2.23), we assume $\tau = 0$. By symmetry, (4.2) and (4.3) are equivalent to the following system of partial differential equations,

$$\kappa_t + \kappa\kappa_x - 2gf_x + \frac{1}{\rho}p_x = 0, \quad -2\zeta f_x + \frac{1}{\rho}p_z = 0, \tag{7.69}$$

$$g_t + \kappa g_x + \zeta\kappa_z = 0, \quad \zeta_t + \zeta_x\kappa - \zeta\kappa_x = 0, \tag{7.70}$$

$$gg_x + 2\zeta g_z - g\zeta_z = 0, \quad \zeta\zeta_z + 2\zeta_x g - g_x\zeta = 0. \tag{7.71}$$

After some computations, we take

$$g = \alpha(t), \quad \zeta = \beta(t) \tag{7.72}$$

for functions α, β of t . So (7.71) naturally holds. By (7.70), we take

$$\kappa = \frac{\beta'x - \alpha'z}{\beta}. \tag{7.73}$$

The compatibility $p_{xz} = p_{zx}$ in (7.69) gives

$$2(\beta\partial_x - \alpha\partial_z)(f_x) = \frac{\alpha''}{\beta}. \tag{7.74}$$

Thus

$$f_x = \frac{\alpha''(\beta x - \alpha z)}{2\beta(\alpha^2 + \beta^2)} + \phi_{\hat{\omega}}(t, \hat{\omega}), \quad \hat{\omega} = \alpha x + \beta z \tag{7.75}$$

for some two-variable function ϕ . Thus

$$p = 2\rho\phi(t, \hat{\omega}) - \frac{\rho\beta''}{2\beta}x^2 + \frac{\rho\alpha''(\alpha x^2 + 2\beta xz - \alpha z^2)}{2(\alpha^2 + \beta^2)} \tag{7.76}$$

modulo the transformation in (2.21).

Recall $\xi = f + \kappa y + gy^2$ and $\eta = \zeta y^2$. By (4.1), we have the following theorem.

Theorem 7.3. *Let α, β be arbitrary function of t and let $\phi(t, \hat{\omega})$ be an arbitrary two-variable function. Then we have the following solution of the three-dimensional classical non-steady boundary layer equations (1.3), (1.4) and (1.5):*

$$u = 2\alpha y + \frac{\beta'x - \alpha'z}{\beta}, \quad w = 2\beta y, \quad v = \frac{\alpha''(\alpha z - \beta x)}{2\beta(\alpha^2 + \beta^2)} - \phi_{\hat{\omega}}(t, \hat{\omega}) - \frac{\beta'y}{\beta} \quad (7.77)$$

and p is given in (7.76), where $\hat{\omega} = \alpha x + \beta z$.

By the transformation in (2.22) and (2.23), we next consider the solutions of (4.2) and (4.3) in the following form:

$$\xi = fy^3 + gy + h, \quad \eta = \kappa y^3 + \tau y^2 + \mu y, \quad (7.78)$$

where $f, g, h, \kappa, \tau, \mu$ are functions of t, x, z with $f \neq 0$. Note

$$\xi_y = 3fy^2 + g, \quad \xi_{yy} = 6fy, \quad \xi_{yyy} = 6f, \quad \xi_{yt} = 3f_t y^2 + g_t, \quad (7.79)$$

$$\xi_{yx} = 3f_x y^2 + g_x, \quad \xi_{yz} = 3f_z y^2 + g_z, \quad \xi_x = f_x y^3 + g_x y + h_x, \quad (7.80)$$

$$\eta_y = 3\kappa y^2 + 2\tau y + \mu, \quad \eta_{yy} = 6\kappa, \quad \eta_{yt} = 3\kappa_t y^2 + 2\tau_t y + \mu_t \quad (7.81)$$

$$\eta_{yx} = 3\kappa_x y^2 + 2\tau_x y + \mu_x, \quad \eta_{yz} = 3\kappa_z y^2 + 2\tau_z y + \mu_z, \quad \eta_z = \kappa_z y^3 + \tau_z y^2 + \mu_z y. \quad (7.82)$$

So we have

$$\begin{aligned} & \xi_y \xi_{yx} - (\xi_x + \eta_z) \xi_{yy} + \eta_y \xi_{yz} \\ = & (3fy^2 + g)(3f_x y^2 + g_x) - 6fy((f_x + \kappa_z)y^3 + \tau_z y^2 + (g_x + \mu_z)y + h_x) \\ & + (3\kappa y^2 + 2\tau y + \mu)(3f_z y^2 + g_z) \\ = & 3(ff_x - 2f\kappa_z + 3f_z\kappa)y^4 + 6(\tau f_z - f\tau_z)y^3 + 3(f_x g - f g_x - 2f\mu_z + \kappa g_z + f_z \mu)y^2 \\ & + (2\tau g_z - 6fh_x)y + \mu g_z + gg_x, \end{aligned} \quad (7.83)$$

$$\begin{aligned} & \xi_y \eta_{yx} - (\xi_x + \eta_z) \eta_{yy} + \eta_y \eta_{yz} \\ = & (3fy^2 + g)(3\kappa_x y^2 + 2\tau_x y + \mu_x) - ((f_x + \kappa_z)y^3 + \tau_z y^2 + (g_x + \mu_z)y + h_x) \\ & \times (6\kappa y + 2\tau) + (3\kappa y^2 + 2\tau y + \mu)(3\kappa_z y^2 + 2\tau_z y + \mu_z) \\ = & 2(3f\tau_x + 2\kappa_z \tau - f_x \tau)y^3 + 2(g\tau_x + \tau_z \mu - \tau g_x - 3\kappa h_x)y + g\mu_x + \mu\mu_z - 2h_x \tau \\ & + 3(3f\kappa_x - 2f_x \kappa + \kappa\kappa_z)y^4 + [3(f\mu_x + \kappa_x g + \kappa_z \mu - 2\kappa g_x - \kappa\mu_z) + 2\tau\tau_z]y^2. \end{aligned} \quad (7.84)$$

Thus the equations (4.2) and (4.3) are implied by the following system of partial differential equations:

$$ff_x - 2f\kappa_z + 3f_z\kappa = 0, \quad 3f\kappa_x - 2f_x\kappa + \kappa\kappa_z = 0, \quad (7.85)$$

$$\tau f_z - f\tau_z = 0, \quad 3f\tau_x + 2\kappa_z\tau - f_x\tau = 0, \quad (7.86)$$

$$f_t + f_xg - fg_x - 2f\mu_z + \kappa g_z + f_z\mu = 0, \quad (7.87)$$

$$3(\kappa_t + f\mu_x + \kappa_xg + \kappa_z\mu - 2\kappa g_x - \kappa\mu_z) + 2\tau\tau_z = 0, \quad (7.88)$$

$$\tau g_z - 3fh_x = 0, \quad \tau_t + g\tau_x + \tau_z\mu - \tau g_x - 3\kappa h_x = 0, \quad (7.89)$$

$$g_t + \mu g_z + gg_x + \frac{1}{\rho}p_x = 6\nu f, \quad \mu_t + g\mu_x + \mu\mu_z - 2h_x\tau + \frac{1}{\rho}p_z = 6\nu\kappa. \quad (7.90)$$

Let γ be a function of t . Again we use the notations $\hat{\omega} = z \cos \gamma - x \sin \gamma$ and $\tilde{\omega} = z \sin \gamma + x \cos \gamma$. We take the following solutions of (7.85):

$$f = \phi(t, \hat{\omega}) \cos \gamma, \quad \kappa = \phi(t, \hat{\omega}) \sin \gamma \quad (7.91)$$

for a two-variable function ϕ . Since $f_x + \kappa_z = 0$, (7.86) is implied by

$$\tau = \alpha \phi(t, \hat{\omega}) \quad (7.92)$$

for a function α of t . Now (7.87) and (7.88) become

$$-\gamma' \phi \sin \gamma + [\phi_t + \phi_{\hat{\omega}}(\mu \cos \gamma - g \sin \gamma) - \phi g_x - 2\phi \mu_z] \cos \gamma + \phi g_z \sin \gamma = 0, \quad (7.93)$$

$$3[(\gamma' + \mu_x)\phi \cos \gamma + (\phi_t + \phi_{\hat{\omega}}(\mu \cos \gamma - g \sin \gamma) - 2\phi g_x - \phi \mu_z) \sin \gamma] + 2\tau\tau_z = 0. \quad (7.94)$$

Moreover $\cos \gamma \times (7.93) - 3 \sin \gamma \times (7.94)$ yields

$$3[\gamma' + \mu_x \cos^2 \gamma - g_z \sin^2 \gamma + (\mu_z - g_x) \sin \gamma \cos \gamma] + 2\alpha^2 \phi_{\hat{\omega}} \cos^2 \gamma = 0, \quad (7.95)$$

equivalently,

$$\partial_{\hat{\omega}}(\mu \cos \gamma - g \sin \gamma) = -\frac{2}{3}\alpha^2 \phi_{\hat{\omega}} \cos^2 \gamma - \gamma'. \quad (7.96)$$

According to (7.91) and (7.92), the first equation in (7.89) gives

$$h_x = \frac{\alpha}{3}g_z \sec \gamma. \quad (7.97)$$

Substituting it into the second equation in (7.89), we get

$$\alpha' \phi + \alpha[\phi_t + \phi_{\hat{\omega}}(\mu \cos \gamma - g \sin \gamma) - \phi \partial_{\hat{\omega}}(g) \sec \gamma] = 0. \quad (7.98)$$

Furthermore, (7.93) can be written as

$$-\gamma' \phi \sin \gamma + [\phi_t + \phi_{\hat{\omega}}(\mu \cos \gamma - g \sin \gamma)] \cos \gamma - \phi \partial_{\hat{\omega}}(g) - 2\phi \partial_z(\mu \cos \gamma - g \sin \gamma) = 0. \quad (7.99)$$

The above two equations implies

$$\partial_z(\cos \gamma \mu - \sin \gamma g) = -\frac{1}{2} \left(\frac{\alpha'}{\alpha} \cos \gamma + \gamma' \sin \gamma \right). \quad (7.100)$$

So

$$\mu \cos \gamma - g \sin \gamma = -\frac{z}{2} \left(\frac{\alpha'}{\alpha} \cos \gamma + \gamma' \sin \gamma \right) + \varepsilon(t, x) \quad (7.101)$$

for some two-variable function ε . Substituting it into (7.96), we obtain

$$-\frac{\sin \gamma}{2} \left(\frac{\alpha'}{\alpha} \cos \gamma + \gamma' \sin \gamma \right) + \varepsilon_x \cos \gamma = -\frac{2}{3} \alpha^2 \phi_{\hat{\omega}} \cos^2 \gamma - \gamma', \quad (7.102)$$

which shows that both ε_x and $\phi_{\hat{\omega}}$ are purely a function of t . Thus we take

$$\phi = \beta \hat{\omega} \quad (7.103)$$

for a function β of t . Hence we have

$$\mu \cos \gamma - g \sin \gamma = -\left(\frac{2}{3} \alpha^2 \beta \cos^2 \gamma + \gamma' \right) \hat{\omega} + \left(\frac{\alpha^2 \beta}{3} \sin 2\gamma + \frac{\gamma'}{2} \tan \gamma - \frac{\alpha'}{2\alpha} \right) \hat{\omega} \quad (7.104)$$

modulo the transformations in (2.17)-(2.20).

In order to solve (7.90) and (7.98), we assume $\alpha = 2$ and

$$\frac{2}{3} \alpha^2 \beta \cos^2 \gamma + 2\gamma' = 0 \implies \beta = -\frac{3\gamma'}{4 \cos^2 \gamma}. \quad (7.105)$$

Then

$$\mu \cos \gamma - g \sin \gamma = \gamma' \hat{\omega} \implies \mu = g \tan \gamma + \gamma' \hat{\omega} \sec \gamma \quad (7.106)$$

by (7.104). Moreover, (7.98) becomes

$$\beta' \hat{\omega} - \beta \hat{\omega} \partial_{\hat{\omega}}(g) \sec \gamma = 0 \implies \partial_{\hat{\omega}}(g) = \frac{\beta'}{\beta} \cos \gamma = \frac{\gamma''}{\gamma'} \cos \gamma + 2\gamma' \sin \gamma. \quad (7.107)$$

Thus

$$g = \left(\frac{\gamma''}{\gamma'} \cos \gamma + 2\gamma' \sin \gamma \right) \hat{\omega} + \psi(t, \hat{\omega}) \quad (7.108)$$

for some two-variable function ψ . The compatibility $p_{xz} = p_{zx}$ in (7.90) becomes

$$(\mu_x - g_z)_t + (g(\mu_x - g_z))_x + (\mu(\mu_x - g_z))_z + 2\gamma'(\hat{\omega} g_z)_x \sec^3 \gamma = -6\nu\beta \quad (7.109)$$

by (7.97) and (7.105).

Observe that (7.106) and (7.108) yield

$$g_z = \frac{\gamma''}{\gamma'} \sin \gamma \cos \gamma + 2\gamma' \sin^2 \gamma + \psi_{\hat{\omega}} \cos \gamma, \quad \mu_x - g_z = \gamma' - \psi_{\hat{\omega}} \sec \gamma. \quad (7.110)$$

So

$$(\mu_x - g_z)_t = \gamma'' - \gamma' \frac{\sin \gamma}{\cos^2 \gamma} \psi_{\hat{\omega}} + \gamma' \tilde{\omega} \psi_{\hat{\omega}\hat{\omega}} \sec \gamma - \psi_{\hat{\omega}t} \sec \gamma, \quad (7.111)$$

$$(g(\mu_x - g_z))_x + (\mu(\mu_x - g_z))_z = \left(\frac{\gamma''}{\gamma'} + 3\gamma' \tan \gamma \right) (\gamma' - \psi_{\hat{\omega}} \sec \gamma) - \gamma' \tilde{\omega} \psi_{\hat{\omega}\hat{\omega}} \sec \gamma \quad (7.112)$$

and

$$(\hat{\omega} g_z)_x = - \left(\frac{\gamma''}{\gamma'} \sin \gamma \cos \gamma + 2\gamma' \sin^2 \gamma + (\psi_{\hat{\omega}} + \hat{\omega} \psi_{\hat{\omega}\hat{\omega}}) \cos \gamma \right) \sin \gamma. \quad (7.113)$$

Thus (7.109) is equivalent to

$$\begin{aligned} & -\psi_{\hat{\omega}t} \sec \gamma - 2\gamma' \hat{\omega} \psi_{\hat{\omega}\hat{\omega}} \tan \gamma \sec \gamma - \left(\frac{\gamma''}{\gamma'} + 6\gamma' \tan \gamma \right) \psi_{\hat{\omega}} \sec \gamma \\ & + 2\gamma''(1 - \tan^2 \gamma) + (\gamma')^2(3 - 4 \tan^2 \gamma) \tan \gamma = \frac{9\nu\gamma'}{2 \cos^2 \gamma}, \end{aligned} \quad (7.114)$$

which can be written as

$$\begin{aligned} & \psi_{\hat{\omega}t} + 2\gamma' \tan \gamma \hat{\omega} \psi_{\hat{\omega}\hat{\omega}} + \left(\frac{\gamma''}{\gamma'} + 6\gamma' \tan \gamma \right) \psi_{\hat{\omega}} \\ & = 2\gamma''(1 - \tan^2 \gamma) \cos \gamma + (\gamma')^2(3 - 4 \tan^2 \gamma) \sin \gamma - \frac{9\nu\gamma'}{2 \cos \gamma}. \end{aligned} \quad (7.115)$$

Hence

$$\begin{aligned} \psi_{\hat{\omega}} &= \frac{\cos^6 \gamma}{\gamma'} \mathfrak{S}'(\hat{\omega} \cos^2 \gamma) + \frac{9\nu}{2} \ln(\sec \gamma - \tan \gamma) \\ &+ \int [2\gamma''(1 - \tan^2 \gamma) \cos \gamma + (\gamma')^2(3 - 4 \tan^2 \gamma) \sin \gamma] dt. \end{aligned} \quad (7.116)$$

for some one-variable function \mathfrak{S} . Therefore,

$$\begin{aligned} \psi &= \frac{\cos^4 \gamma}{\gamma'} \mathfrak{S}(\hat{\omega} \cos^2 \gamma) + \frac{9\nu}{2} \hat{\omega} \ln(\sec \gamma - \tan \gamma) + \varphi \\ &+ \hat{\omega} \int [2\gamma''(1 - \tan^2 \gamma) \cos \gamma + (\gamma')^2(3 - 4 \tan^2 \gamma) \sin \gamma] dt \end{aligned} \quad (7.117)$$

for some function φ of t . According to (7.108),

$$\begin{aligned} g &= \left(\frac{\gamma''}{\gamma'} \cos \gamma + 2\gamma' \sin \gamma \right) \tilde{\omega} + \frac{\cos^4 \gamma}{\gamma'} \mathfrak{S}(\hat{\omega} \cos^2 \gamma) + \frac{9\nu}{2} \hat{\omega} \ln(\sec \gamma - \tan \gamma) + \varphi \\ &+ \hat{\omega} \int [2\gamma''(1 - \tan^2 \gamma) \cos \gamma + (\gamma')^2(3 - 4 \tan^2 \gamma) \sin \gamma] dt, \end{aligned} \quad (7.118)$$

$$\begin{aligned}\mu &= \left[\frac{\cos^4 \gamma}{\gamma'} \Im(\hat{\omega} \cos^2 \gamma) + \hat{\omega} \int [2\gamma''(1 - \tan^2 \gamma) \cos \gamma + (\gamma')^2(3 - 4 \tan^2 \gamma) \sin \gamma] dt \right. \\ &\quad \left. + \frac{9\nu}{2} \hat{\omega} \ln(\sec \gamma - \tan \gamma) + \varphi \right] \tan \gamma + \left(\frac{\gamma''}{\gamma'} \sin \gamma + \gamma' \frac{1 + 2 \sin^2 \gamma}{\cos \gamma} \right) \hat{\omega},\end{aligned}\quad (7.119)$$

$$\begin{aligned}h_x &= 3\nu \ln(\sec \gamma - \tan \gamma) + \frac{2}{3} \left\{ \int [2\gamma''(1 - \tan^2 \gamma) \cos \gamma + (\gamma')^2(3 - 4 \tan^2 \gamma) \sin \gamma] dt \right. \\ &\quad \left. + \left(\frac{\gamma''}{\gamma'} + 2\gamma' \tan \gamma \right) \tan \gamma + \frac{\cos^6 \gamma}{\gamma'} \Im'(\hat{\omega} \cos^2 \gamma) \right\}.\end{aligned}\quad (7.120)$$

According to (7.91), (7.92), (7.97), (7.103), (7.105), (7.106) and (7.108),

$$\begin{aligned}&g_t + \mu g_z + g g_x - 6\nu f \\ &= \left(\gamma'' \sin \gamma + \frac{\gamma'' \gamma' - (\gamma'')^2 + 2(\gamma')^4}{(\gamma')^2} \cos \gamma \right) \hat{\omega} + (\gamma'' \cos \gamma + 2(\gamma')^2 \sin \gamma) \hat{\omega} + \psi_t \\ &\quad - \gamma' \psi_{\hat{\omega}} \hat{\omega} + \left(\frac{\gamma''}{\gamma'} \cos \gamma + 2\gamma' \sin \gamma \right)^2 \hat{\omega} \sec \gamma + \left(\frac{\gamma''}{\gamma'} \cos \gamma + 2\gamma' \sin \gamma \right) \psi \sec \gamma \\ &\quad + (\gamma'' \cos \gamma + 2(\gamma')^2 \sin \gamma) \hat{\omega} \tan \gamma + \gamma' \psi_{\hat{\omega}} \hat{\omega} + \frac{9\nu \gamma'}{2 \cos \gamma} \hat{\omega} \\ &= \left(6(\gamma'' + (\gamma')^2 \tan \gamma) \sin \gamma + \frac{\gamma'' \gamma' + 2(\gamma')^4}{(\gamma')^2} \cos \gamma \right) \hat{\omega} + \left(\frac{\gamma''}{\gamma'} + 2\gamma' \tan \gamma \right) \psi \\ &\quad + \psi_t + (\gamma'' \cos \gamma + 2(\gamma')^2 \sin \gamma) \hat{\omega} + \frac{9\nu \gamma'}{2 \cos \gamma} \hat{\omega},\end{aligned}\quad (7.121)$$

$$\begin{aligned}&\mu_t + g \mu_x + \mu \mu_z - 2h_x \tau - 6\nu \kappa \\ &= 2\gamma' g \sec^2 \gamma + \gamma' \hat{\omega} \sec \gamma + 2(\gamma')^2 \hat{\omega} \tan \gamma \sec \gamma + (\gamma')^2 \hat{\omega} \sec \gamma \\ &\quad + (g_t + g g_x + \mu g_z - 6\nu f) \tan \gamma + 2\gamma' g_z \sec^3 \gamma \\ &= 2[(\gamma'' \cos \gamma + 2(\gamma')^2 \sin \gamma) \hat{\omega} + \gamma' \psi] \sec^2 \gamma + \gamma'' \hat{\omega} \sec \gamma + 2(\gamma')^2 \hat{\omega} \tan \gamma \sec \gamma \\ &\quad + (\gamma')^2 \hat{\omega} \sec \gamma + 2[(\gamma'' \cos \gamma + 2(\gamma')^2 \sin \gamma) \sin \gamma + \gamma' \psi_{\hat{\omega}} \cos \gamma] \hat{\omega} \sec^3 \gamma \\ &\quad + (g_t + g g_x + \mu g_z - 6\nu f) \tan \gamma \\ &= 3(\gamma'' + 2(\gamma')^2 \tan \gamma) \hat{\omega} \sec \gamma + ((\gamma')^2 + 2\gamma'' \tan \gamma + 4(\gamma')^2 \tan^2 \gamma) \hat{\omega} \sec \gamma \\ &\quad + 2\gamma' (\psi + \hat{\omega} \psi_{\hat{\omega}}) \sec^2 \gamma + (g_t + g g_x + \mu g_z - 6\nu f) \tan \gamma.\end{aligned}\quad (7.122)$$

Expression (7.115) says

$$\begin{aligned}&\left(\frac{\gamma''}{\gamma'} + 2\gamma' \tan \gamma \right) \psi + \psi_t + (\gamma'' \cos \gamma + 2(\gamma')^2 \sin \gamma) \hat{\omega} + \frac{9\nu \gamma'}{2 \cos \gamma} \hat{\omega} \\ &= \varphi' + [\gamma''(3 - 2 \tan^2 \gamma) \cos \gamma + (\gamma')^2 \sin \gamma (5 - 4 \tan^2 \gamma)] \hat{\omega} \\ &\quad - 2\gamma' \hat{\omega} \psi_{\hat{\omega}} \tan \gamma - 2\gamma' \psi \tan \gamma + \left(\frac{\gamma''}{\gamma'} + 4\gamma' \tan \gamma \right) \varphi.\end{aligned}\quad (7.123)$$

By (7.121) and (7.123), we set

$$\begin{aligned}\hat{g} &= \frac{\tilde{\omega}^2}{2} \left(6 \sin \gamma (\gamma'' + (\gamma')^2 \tan \gamma) + \frac{\gamma'' \gamma' + 2(\gamma')^4}{(\gamma')^2} \cos \gamma \right) \sec \gamma \\ &\quad + \frac{\hat{\omega}^2}{2} [\gamma'' (2 \tan \gamma - 3 \cot \gamma) + (\gamma')^2 (4 \tan^2 \gamma - 5)] \\ &\quad + 2\gamma' \hat{\omega} \psi \sec \gamma + \left[\varphi' + \left(\frac{\gamma''}{\gamma'} + 4\gamma' \tan \gamma \right) \varphi \right] \tilde{\omega} \sec \gamma.\end{aligned}\quad (7.124)$$

Then

$$\hat{g}_x = g_t + \mu g_z + g g_x - 6\nu f \quad (7.125)$$

by (4.31). Moreover, (4.31) yields $\partial_z(F(t, \hat{\omega})) = -\partial_x(F(t, \hat{\omega})) \cot \gamma$ for any function F of t and $\hat{\omega}$. Furthermore, (7.122), (7.124) and (7.126) imply

$$\begin{aligned}&\mu_t + g\mu_x + \mu\mu_z - 2h_x\tau - 6\nu\kappa - \hat{g}_z \\ &= 3(\gamma'' + 2(\gamma')^2 \tan \gamma) \tilde{\omega} \sec \gamma + ((\gamma')^2 + 2\gamma'' \tan \gamma + 4(\gamma')^2 \tan^2 \gamma) \hat{\omega} \sec \gamma \\ &\quad + 2\gamma'(\psi + \hat{\omega}\psi_{\hat{\omega}}) \sec^2 \gamma + (\tan \gamma + \cot \gamma) \{ [\gamma''(3 - 2 \tan^2 \gamma) \cos \gamma \\ &\quad + (\gamma')^2(5 - 4 \tan^2 \gamma) \sin \gamma] \hat{\omega} - 2\gamma' \hat{\omega} \psi_{\hat{\omega}} \tan \gamma - 2\gamma' \psi \tan \gamma \} \\ &= 3(\gamma'' + 2(\gamma')^2 \tan \gamma) \tilde{\omega} \sec \gamma + ((\gamma')^2 + 2\gamma'' \tan \gamma + 4(\gamma')^2 \tan^2 \gamma) \hat{\omega} \sec \gamma \\ &\quad + [\gamma''(3 - 2 \tan^2 \gamma) \csc \gamma + (\gamma')^2(5 - 4 \tan^2 \gamma) \sec \gamma] \hat{\omega} \\ &= 6z(\csc 2\gamma + \sec^2 \gamma).\end{aligned}\quad (7.126)$$

Therefore,

$$\begin{aligned}p &= -\rho \{ 3z^2(\csc 2\gamma + \sec^2 \gamma) + \left[\varphi' + \left(\frac{\gamma''}{\gamma'} + 4\gamma' \tan \gamma \right) \varphi \right] \tilde{\omega} \sec \gamma \\ &\quad + 2\gamma' \hat{\omega} \psi \sec \gamma + \frac{\hat{\omega}^2}{2} [\gamma'' (2 \tan \gamma - 3 \cot \gamma) + (\gamma')^2 (4 \tan^2 \gamma - 5)] \\ &\quad + \frac{\tilde{\omega}^2}{2} \left(6(\gamma'' + (\gamma')^2 \tan \gamma) \sin \gamma + \frac{\gamma'' \gamma' + 2(\gamma')^4}{(\gamma')^2} \cos \gamma \right) \sec \gamma \}\end{aligned}\quad (7.127)$$

modulo the transformation in (2.21).

By (4.1) and (7.78), we have the following theorem.

Theorem 7.4. *Let γ, φ be arbitrary functions of t and let \mathfrak{S} be an arbitrary one-variable function. Denote $\hat{\omega} = z \cos \gamma - x \sin \gamma$ and $\tilde{\omega} = z \sin \gamma + x \cos \gamma$. We have the following solutions of the three-dimensional classical non-steady boundary layer equations (1.3), (1.4) and (1.5):*

$$\begin{aligned}u &= \left(\frac{\gamma''}{\gamma'} \cos \gamma + 2\gamma' \sin \gamma \right) \tilde{\omega} + \frac{\cos^4 \gamma}{\gamma'} \mathfrak{S}(\hat{\omega} \cos^2 \gamma) + \frac{9\nu}{2} \hat{\omega} \ln(\sec \gamma - \tan \gamma) + \varphi \\ &\quad + \hat{\omega} \int [2\gamma''(1 - \tan^2 \gamma) \cos \gamma + (\gamma')^2(3 - 3 \tan^2 \gamma) \sin \gamma] dt - \frac{9\gamma' \hat{\omega}}{4 \cos \gamma} y^2,\end{aligned}\quad (7.128)$$

$$\begin{aligned}
w = & \left[\frac{\cos^4 \gamma}{\gamma'} \Im(\hat{\omega} \cos^2 \gamma) + \hat{\omega} \int [2\gamma''(1 - \tan^2 \gamma) \cos \gamma + (\gamma')^2(3 - 4 \tan^2 \gamma) \sin \gamma] dt \right. \\
& + \frac{9\nu}{2} \hat{\omega} \ln(\sec \gamma - \tan \gamma) + \varphi] \tan \gamma + \left(\frac{\gamma''}{\gamma'} \sin \gamma + \gamma' \frac{1 + 2 \sin^2 \gamma}{\cos \gamma} \right) \hat{\omega} \\
& \left. - 3\gamma' \hat{\omega} y \sec^2 \gamma - \frac{9\gamma' \hat{\omega} \sin \gamma}{4 \cos^2 \gamma} y^2, \right. \tag{7.129}
\end{aligned}$$

$$\begin{aligned}
v = & \frac{3}{2} \gamma' y^2 \sec \gamma - \left(\frac{\gamma''}{\gamma'} + 3\gamma' \tan \gamma \right) y - 3\nu \ln(\sec \gamma - \tan \gamma) \\
& - \frac{2}{3} \left\{ \int [2\gamma''(1 - \tan^2 \gamma) \cos \gamma + (\gamma')^2(3 - 4 \tan^2 \gamma) \sin \gamma] dt \right. \\
& \left. + \left(\frac{\gamma''}{\gamma'} + 2\gamma' \tan \gamma \right) \tan \gamma + \frac{\cos^6 \gamma}{\gamma'} \Im(\hat{\omega} \cos \gamma) \right\}. \tag{7.130}
\end{aligned}$$

and p is given (7.127) via (7.117).

Finally, we consider

$$\xi = fy^3 + gy^2 + \kappa y + h, \quad \eta = \tau y + \zeta y^{-1}, \tag{7.131}$$

where f, g, h, κ, τ and ζ are functions of t, x, z with $\zeta \neq 0$. First,

$$\xi_y = 3fy^2 + 2gy + \kappa, \quad \xi_{yy} = 6fy + 2g, \quad \xi_{yyy} = 6f, \tag{7.132}$$

$$\xi_{yt} = 3f_t y^2 + 2g_t y + \kappa_t, \quad \xi_{yx} = 3f_x y^2 + 2g_x y + \kappa_x, \tag{7.133}$$

$$\xi_{yz} = 3f_z y^2 + 2g_z y + \kappa_z, \quad \xi_x = f_x y^3 + g_x y^2 + \kappa_x y + h_x, \tag{7.134}$$

$$\eta_y = \tau - \zeta y^{-2}, \quad \eta_{yy} = 2\zeta y^{-3}, \quad \eta_{yyy} = -6\zeta y^{-4}, \quad \eta_{yx} = \tau_x - \zeta_x y^{-2} \tag{7.135}$$

$$\eta_{yt} = \tau_t - \zeta_t y^{-2}, \quad \eta_{yz} = \tau_z - \zeta_z y^{-2}, \quad \eta_z = \tau_z y + \zeta_z y^{-1}. \tag{7.136}$$

So we have

$$\begin{aligned}
& \xi_y \xi_{yx} - (\xi_x + \eta_z) \xi_{yy} + \eta_y \xi_{yz} = (3fy^2 + 2gy + \kappa)(3f_x y^2 + 2g_x y + \kappa_x) \\
& - (6fy + 2g)(f_x y^3 + g_x y^2 + (\kappa_x + \tau_z)y + h_x + \zeta_z y^{-1}) + (\tau - \zeta y^{-2})(3f_z y^2 + 2g_z y + \kappa_z) \\
& = 3ff_x y^4 + 4fg_x y^3 + (3f_x \kappa + 2gg_x - 3f\kappa_x - 6f\tau_z + 3f_z \tau) y^2 - 2(g\zeta_z + g_z \zeta) y^{-1} \\
& + 2(g_x \kappa - 3fh_x - g\tau_z + g_z \tau) y + \kappa \kappa_x - 6f\zeta_z - 2gh_x + \tau \kappa_z - 3f_z \zeta - \zeta \kappa_z y^{-2}, \tag{7.137}
\end{aligned}$$

$$\begin{aligned}
& \xi_y \eta_{yx} - (\xi_x + \eta_z) \eta_{yy} + \eta_y \eta_{yz} = (3fy^2 + 2gy + \kappa)(\tau_x - \zeta_x y^{-2}) \\
& - 2\zeta y^{-3}(f_x y^3 + g_x y^2 + (\kappa_x + \tau_z)y + h_x + \zeta_z y^{-1}) + (\tau - \zeta y^{-2})(\tau_z - \zeta_z y^{-2}) \\
& = 3f\tau_x y^2 + 2g\tau_x y + \kappa \tau_x + \tau \tau_z - 3f\zeta_x - 2f_x \zeta - 2(g\zeta)_x y^{-1} \\
& - (\kappa \zeta_x + 2\zeta \kappa_x + 3\zeta \tau_z + \tau \zeta_z) y^{-2} - 2h_x \zeta y^{-3} - \zeta \zeta_z y^{-4}. \tag{7.138}
\end{aligned}$$

Hence (4.2) and (4.3) are implied by the following system of partial differential equations:

$$f_x = (g\zeta)_z = \kappa_z = \tau_x = (g\zeta)_x = h_x = 0, \quad \zeta_z = 6\nu, \quad (7.139)$$

$$f_t + \frac{2}{3}gg_x - f\kappa_x - 2f\tau_z + f_z\tau = 0, \quad (7.140)$$

$$g_t + g_x\kappa - g\tau_z + g_z\tau = 0, \quad (7.141)$$

$$\kappa_t + \kappa\kappa_x - 6f\zeta_z - 3f_z\zeta + \frac{1}{\rho}p_x = 6\nu f, \quad (7.142)$$

$$\tau_t + \tau\tau_z - 3f\zeta_x + \frac{1}{\rho}p_z = 0, \quad (7.143)$$

$$\zeta_t + \kappa\zeta_x + 2\zeta\kappa_x + 3\zeta\tau_z + \tau\zeta_z = 0. \quad (7.144)$$

By (7.139), we take

$$h = 0, \quad f = \phi(t, z), \quad \zeta = 6\nu z + \psi(t, x), \quad g = \alpha\zeta^{-1} \quad (7.145)$$

for some two-variable functions ϕ, ψ and a function α of t . Moreover, (7.144) becomes

$$\psi_t + \kappa\psi_x + 2\psi\kappa_x + 12\nu z\kappa_x + 3\psi\tau_z + 18\nu z\tau_z + 6\nu\tau = 0. \quad (7.146)$$

Modulo the transformation in (2.19) and (2.20), we assume $\psi = 0$ or $\psi_x \neq 0$. Note the compatibility $p_{xz} = p_{zx}$ in (7.142) and (7.143) gives

$$f\zeta_{xx} = 14\nu f_z + \zeta f_{zz} \implies f\psi_{xx} - \psi f_{zz} = 20\nu f_z + 6\nu z f_{zz}. \quad (7.147)$$

Furthermore, the last equation in (7.145) says that $g \times (7.144) + \zeta \times (7.141)$ yields

$$\alpha' + 2\alpha(\kappa_x + \tau_z) = 0 \quad (7.148)$$

Subcase 1. $\psi_x \neq 0$.

In this subcase, applying $\partial_x \partial_z$ to (7.146) gives

$$4\nu\kappa_{xx} + \psi_x\tau_{zz} = 0. \quad (7.149)$$

So τ is a quadratic polynomial in z . The coefficients of z^2 in (7.146) say that $\tau_{zz} = 0$, and so $\kappa_{xx} = 0$. By the coefficients of z in (7.146), we have

$$\kappa = -2\beta'x, \quad \tau = \beta'z \quad (7.150)$$

for some function β of t modulo the transformations in (2.17)-(2.20). Now (7.146) becomes

$$\psi_t - 2\beta'x\psi_x - \beta'\psi = 0. \quad (7.151)$$

Hence

$$\psi = e^\beta \mathfrak{S}(e^{2\beta} x) \quad (7.152)$$

for some one-variable function \mathfrak{S} . According to (7.147), we take $f = 0$. Furthermore, (7.140) gives us $g = 0$.

Subcase 2. $\psi = 0$.

In this subcase, (7.146) becomes

$$2z\kappa_x + 3z\tau_z + \tau = 0. \quad (7.153)$$

So

$$\kappa_x = -2\beta', \quad \tau = \gamma z^{-1/3} + \beta' z \quad (7.154)$$

for some functions β and γ . Moreover, (7.148) yields

$$\alpha = a\delta_{\gamma,0}e^{2\beta} \quad a \in \mathbb{R}. \quad (7.155)$$

Now (7.140) becomes

$$f_t + \frac{2\gamma}{3}z^{-4/3}f + (\gamma z^{-1/3} + \beta' z)f_z = 0 \quad (7.156)$$

and (7.147) gives

$$10f_z + 3zf_{zz} = 0. \quad (7.157)$$

These two equations force us to take $\gamma = 0$ and

$$f = be^{7\beta/3}z^{-7z/3} + c, \quad b, c \in \mathbb{R}. \quad (7.158)$$

By (4.1) and (7.131), we have the following theorem:

Theorem 7.5. *Let β be arbitrary function of t and let a, b, c be any real constants. Suppose that \mathfrak{S} is any one-variable function. We have the following solutions of the three-dimensional classical non-steady boundary layer equations (1.3), (1.4) and (1.5): (1)*

$$u = -2\beta'x, \quad v = \beta'y - 6\nu y^{-1}, \quad w = \beta'z - [6\nu z + e^\beta \mathfrak{S}(e^{2\beta} x)]y^{-2}, \quad (7.159)$$

$$p = \rho(\beta'' - 2(\beta')^2)x^2 - \frac{\rho}{2}(\beta'' + (\beta')^2)z^2. \quad (7.160)$$

(2)

$$u = 3(be^{7\beta/3}z^{-7z/3} + c)y^2 + \frac{ae^{2\beta}y}{3\nu z} - 2\beta'x, \quad v = \beta'y - 6\nu y^{-1}, \quad w = (\beta' - 6\nu y^{-2})z, \quad (7.161)$$

$$p = \rho[42\nu cx + (\beta'' - 2(\beta')^2)x^2] - \frac{\rho}{2}(\beta'' + (\beta')^2)z^2. \quad (7.162)$$

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